Automata and Formal Languages. Homework 9 (11.11.2024)

1. Let L be the language generated by the grammar $(\{S\}, \{a, b\}, P, S)$ where P contains the productions $S \longrightarrow aSSa \mid SbbS \mid \varepsilon$. Moreover, let $h : \{a, b\}^* \longrightarrow \{a, b\}^*$ be the homomorphism defined by $h(a) = ab$ and $h(b) = ba$. Construct a context-free grammar that generates the language $h(L^*) \cup LL$.

2. Is the family of context-free languages closed under the following language operations ? Prove.

   (a) Palindrome$(L) = \{w \mid w \in L$ and w is a palindrome $\}$.

   (b) $L^R = \{w \mid w^R \in L\}$.

   (c) Half$(L) = \{w \mid ww \in L\}$.

   (d) Pref$(L) = \{w \mid wu \in L$ for some word $u\ \}$.

3. Let $L = \{a^n b^m c^k \mid\ m \neq n$ or $m \neq k\ \}$. Prove that L is context-free but its complement is not context-free.

4. Prove that $L = b^*c^*d^* \cup \{a^n b^m c^m d^m \mid n, m \geq 0\}$ is a non-context-free language that satisfies the pumping lemma of context-free languages.

5. Let
$$\begin{aligned} L_1 &= \{a^{2n} b^n \mid n \geq 1\}^* \text{ and} \\ L_2 &= \{b^n a^n \mid n \geq 1\}^* b. \end{aligned}$$
   (Note the Kleene closure.) Prove that L_1 and L_2 are context-free but their quotient L_1/L_2 is not context-free.

6. For any language $L \subseteq \Sigma^*$, we define
$$\text{Shuffle}(L) = \{w \in \Sigma^* \mid\ \exists u \in L :\ w \text{ is obtained from } u \text{ by permuting its letters } \}$$
   to be the language of words that can be obtained by re-ordering the letters in the words of L. (Recall Problem 3 in homework set #4.)

   (a) Prove that there is a regular language L over the three letter alphabet $\{a, b, c\}$ such that Shuffle(L) is not context free.

   (b) Prove that if $\Sigma = \{a, b\}$ and L is regular then Shuffle(L) is context-free.

7. Show that there are algorithms that solve the following problems. (Hint: use effective closure properties and known algorithms from the lecture notes)

   (a) Does a given context-free grammar generate a word that contains aba as a subword ?

   (b) Does every word generated by a given context-free grammar contain subword aba ?

   (c) Does a given regular language contain infinitely many palindromes ?