Automata and Formal Languages. Homework 13 (9.12.2024)

1. Do the following instances to PCP have a solution ? Prove.

   (a) $\begin{cases} L_1 &= a, \quad ba, \quad bb, \\ L_2 &= ab, \quad b, \quad ba. \end{cases}$

   (b) $\begin{cases} L_1 &= ab, \quad bb, \quad aaa, \\ L_2 &= abb, \quad baa, \quad aa. \end{cases}$

2. For any PCP instance

$$\begin{array}{lllll} L_1 & : & w_1, & w_2, & \dots & w_k \\ L_2 & : & x_1, & x_2, & \dots & x_k \end{array}$$

   the *equality set*

$$Eq(L_1, L_2) = \{i_1 i_2 \dots i_m \mid w_{i_1} w_{i_2} \dots w_{i_m} = x_{i_1} x_{i_2} \dots x_{i_m}\}$$

   of the instance is the language over the alphabet $\{1, 2, \dots, k\}$ that contains the empty word ε and all those non-empty index sequences that are a solution to the PCP instance. Show that the PCP instances of Problem 1 have regular equality sets, and construct them.

   (In general, the equality set does not need to be regular, or even context-free. It is, however, always clearly recursive, and even context-sensitive.)

3. Prove that the following questions concerning a given context-free language L are undecidable (See problem 2 in the homework problem set #5 for the definitions of the concepts):

   (a) Is L prefix-free ?

   (b) Is L prefix-closed ?

   (c) Does L contain any square ww ?

   (Hint. Reduce the undecidable problem "Is $L_1 \cap L_2 = \emptyset$?" in (a) and (c). For (b), the undecidable problem "Is $L = \Sigma^*$?" can conveniently be used.)

4. Determine if the following decision problems concerning a given context-free language L are decidable:

   (a) Does L contain at least one palindrome ?

   (b) Is $L = L^R$?

5. Prove that it is undecidable if a given context-free language L is a code. (See problem 4 in the homework set #1 for the definition of a code.)

6. Determine if the following pair of 2×2 matrices is mortal:

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 100 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

7. Let $A_1, \dots, A_6$ be 3×3 square matrices, and let us denote by I and 0 the 3×3 identity and zero matrices, respectively. Prove that the pair

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & 0 & 0 & A_5 \\ 0 & 0 & 0 & 0 & 0 & A_6 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

of 18×18 matrices (presented in block form) is mortal if and only if $\{A_1, \ldots, A_6\}$ is mortal. Hint: It is enough to consider products of matrices of form MP^k.

(Since the mortality is undecidable for given six integer matrices of size 3×3, we can conclude the mortality to be undecidable among pairs of 18×18 matrices.)