Symbolic dynamics: Homework 5 (24.2.2025)

1. Let $A = \{0, 1\}$ and consider the one-sided shift space $A^{\mathbb{N}}$ under the metric defined in the class. Define the function $f : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ that (1) flips all bits of $x \in A^{\mathbb{N}}$ iff $x_0 = 1$ and (2) applies the left shift to the result. In other words, for all $x \in A^{\mathbb{N}}$ and all $k \in \mathbb{N}$,

$$f(x)_k = x_0 + x_{k+1} \pmod 2.$$

   (a) Prove that f is continuous so that $(A^{\mathbb{N}}, f)$ is a dynamical system.

   (b) Prove that for all $x \in A^{\mathbb{N}}$, $n \geq 1$ and $k \in \mathbb{N}$ holds:

$$f^n(x)_k = x_{n-1} + \sigma^n(x)_k \pmod 2.$$

   (In other words, $f^n(x)$ is x shifted left n times, followed by flipping all bits iff $x_{n-1} = 1$.)

2. Let A and f be as in Problem 1 above.

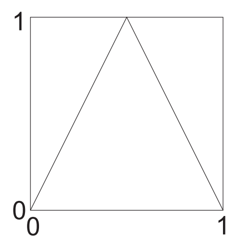
   (a) Prove that $(A^{\mathbb{N}}, f)$ is mixing.

   (b) Prove that the set of periodic points is dense in $A^{\mathbb{N}}$.

3. Consider the dynamical system $(\mathbb{I}, g)$ where $\mathbb{I} = [0, 1]$ is the unit interval with the usual metric and

$$g(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

   is the *tent map*:



   (a) Prove that the system is a factor of the system $(A^{\mathbb{N}}, f)$ of Problem 1 above. (Hint: Consider the map $x \mapsto (x)_2$ of Example 2.11 in the notes.)

   (b) Prove that $(\mathbb{I}, g)$ is mixing and that the set of periodic points is dense in $\mathbb{I}$.

4. Let $A = \{0, 1\}$ and consider the one-sided shift space $A^{\mathbb{N}}$ under the metric defined in the class. Define the *odometer* $f : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ as follows:

$$\begin{aligned} f(1^n 0 u) &= 0^n 1 u, \quad \text{for all } n \in \mathbb{N}, u \in A^{\mathbb{N}}, \\ f(1^\infty) &= 0^\infty. \end{aligned}$$

   (a) Prove that f is an isometry and a homeomorphism.

   (b) Prove that the system is minimal and that every point is quasi-periodic.

5. Let X be a compact metric space and $f : X \longrightarrow X$ an isometry. Consider the dynamical system (X, f) over monoid $\mathbb{N}$.

   (a) Prove that every $x \in X$ is recurrent.

   (b) Prove that f is a homeomorphism.

6. Let (X, f) be a transitive dynamical system over the monoid $(\mathbb{N}, +)$. Suppose the system has an equicontinuity point. Prove:

   (a) A point $x \in X$ is transitive if and only if it is an equicontinuity point.

   (b) System (X, f) is equicontinuous if and only if it is minimal.

7. Let (X, f) be a transitive dynamical system over the monoid $(\mathbb{N}, +)$. Suppose the system has a dense set of periodic points. Prove:

   (a) Every equicontinuity point is quasi-periodic.

   (b) The system is sensitive or finite.