Symbolic dynamics: Homework 1 (27.1.2025)

1. Let $X = \{0, 1, 2, \ldots\}$, and let $\mathcal{T} = \{\emptyset\} \cup \{U_0, U_1, U_2, \ldots\}$ where for every i

$$U_i = \{i, i+1, i+2, \ldots\}.$$

   (a) Show that $\mathcal{T}$ is a topology of X.

   (b) Is $\mathcal{T}$ compact ? Is $\mathcal{T}$ metric ?

   (c) Which subsets of X are compact ?

   (d) Determine which sequences of elements of X converge to 0, and determine also which sequences converge to 1.

2. Let X be a set with a topology $\mathcal{T}$. Prove the following three properties of closed sets (i.e., complements of open sets). Note that these properties are the duals of the three axioms of open sets.

   (a) The empty set $\emptyset$ is closed, and X is closed,

   (b) the intersection of any number of closed sets is closed, and

   (c) the union of a finite number of closed sets is closed.

3. Prove the following properties of the topological closure $\overline{A}$ of $A \subseteq X$.

   (a) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$,

   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

   (c) $\overline{\overline{A}} = \overline{A}$.

4. Let $\mathcal{T}$ be a topology on X, and let $A \subseteq X$ be a non-empty subset. The induced topology on A is defined as $\{A \cap U \mid U \in \mathcal{T}\}$.

   (a) Prove that the induced topology is a topology on A.

   (b) Prove that the closed sets of the induced topology are exactly the sets sets $A \cap F$ where $F \subseteq X$ is closed under $\mathcal{T}$.

   (c) The closure of $S \subseteq A$ under the induced topology is $\overline{S} \cap A$ where $\overline{S}$ is the closure of S under $\mathcal{T}$.

   (d) $B \subseteq A$ is compact under induced topology if and only $B \subseteq A$ is compact under $\mathcal{T}$.

5. (a) Prove that in a metric space X, every compact $A \subseteq X$ is closed and bounded. (Bounded means that A is contained in some ε-ball.)

   (b) Prove that the converse is not true: there is a metric space where a closed and bounded set is not necessarily compact.

6. Prove that an open subset of a Baire space is also a Baire space (under the induced topology).

7. Let X be a compact space, and suppose the topology has a base all of whose members are clopen (closed and open). Prove that a set is clopen if and only if it is a finite union of base sets.