Introduction

A discrete time topological dynamical system is a **continuous transformation**

$$f : X \longrightarrow X$$

on a **compact metric space** X. Iterating the system from an initial point $x \in X$ yields a trajectory

$$x \mapsto f(x) \mapsto f^2(x) \mapsto \dots$$

One is typically interested in how sensitive the system is small changes in the initial point x, or in mixing properties of the system, or in identifying attractors or subsystems, etc.

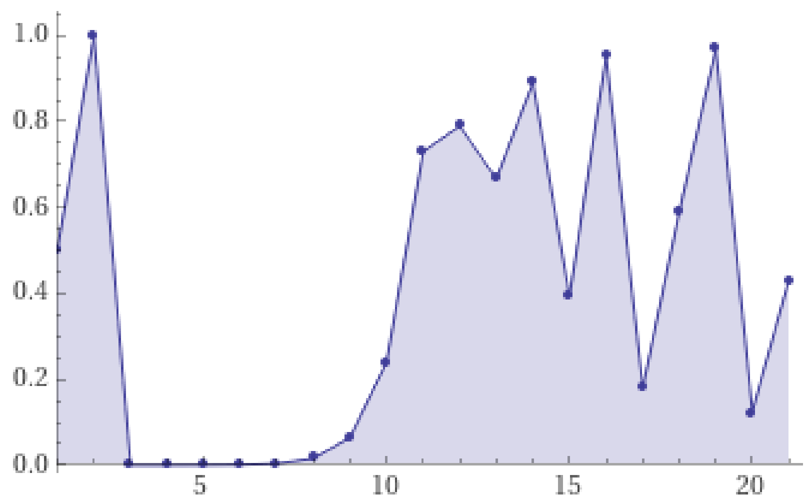
Example. Let $X = [0,1]$ and $f(x) = 4x(1-x)$, the so-called Logistic map.

An initial point $x = 0.5$ has the trajectory

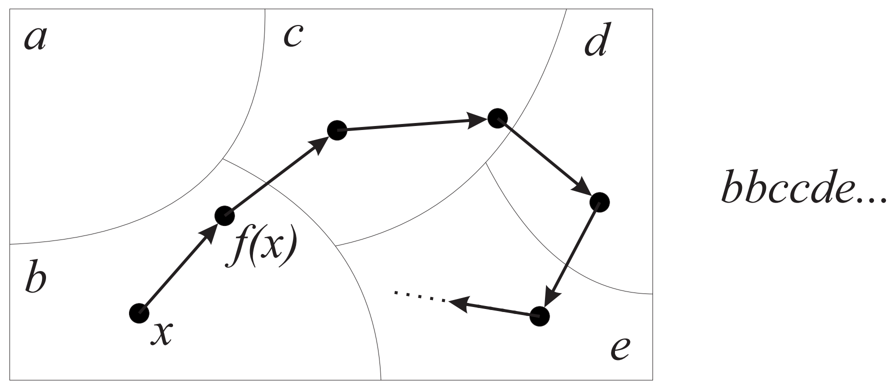
$$0.5 \mapsto 1 \mapsto 0 \mapsto 0 \ldots$$

that leads to a fixed point in two steps. The system is **chaotic** in the sense that a tiny change in the initial point may get magnified and lead to a totally different trajectory (butterfly effect).

Here are the first 20 iterates from $x = 0.501$:



An **observer** can only observe a system up to a certain precision $\Longrightarrow$ space X is partitioned into a finite number of mutually distinguishable parts:



The observed trajectories are then infinite words (=sequences of symbols), representing the observations at times $t = 0, 1, 2, \ldots$

Some sequence may be impossible, e.g., maybe there is no point whose trajectory reads $acd\ldots$. Then word acd does not appear in any valid sequence. This leads to **subshifts**: sets of sequences obtained by forbidding some finite words.

Different types on subshifts based on the complexity of the forbidden set:

- finitely many forbidden words $\Longrightarrow$ **subshift of finite type**
- regular language forbidden $\Longrightarrow$ **sofic subshift**
- RE language forbidden $\Longrightarrow$ **effective subshift**

If f is bijective then one can also consider two-way infinite trajectories

$$\ldots f^{-2}(x), f^{-1}(x), x, f(x), f^2(x) \ldots,$$

which leads to bi-infinite words and two-sided subshifts.

More generally, we may consider dynamical systems with several transformations $f_1, \ldots, f_n$ that can be applied in an arbitrary order.

These topics are studied in the field of **topological dynamics**. Everything hinges on f being a continuous map of a compact metric space. Thus we begin with a short review of metric spaces, topology, compactness and continuity.

Symbolic dynamics is a sub-field of topological dynamics where the considered systems are shifts on sets of infinite words (or infinite **configurations**). These sets are given a suitable metric that makes them into compact spaces, and thus the general results of topological dynamics can be applied.

Notations

- The natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ contain 0.

- Composition of functions is from right-to-left so

$$(g \circ f)(x) = g(f(x)).$$

- For sets A and B, we denote by B^A the set of functions

$$f : A \longrightarrow B.$$

- The restriction of function $f : A \longrightarrow B$ to subset $S \subseteq A$ is denoted by $f_{|S}$.

- For a function $f : A \longrightarrow B$ any $S \subseteq A$ and $T \subseteq B$, we denote

$$\begin{aligned} f^{-1}(T) &= \{a \in A \mid f(a) \in T\}, \\ f(S) &= \{f(s) \mid s \in S\}. \end{aligned}$$

- $\mathbf{id}_A$ the identity function $A \longrightarrow A$.

Metric spaces and topology

The usual Euclidean distance d on $\mathbb{R}^k$ has the following properties: For all $x, y, z \in \mathbb{R}^k$

(a) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$; (positivity),

(b) $d(x, y) = d(y, x)$; (symmetry) and

(c) $d(x, y) \leq d(x, z) + d(z, y)$; (the triangle inequality).

Many properties of the space can be proved using these properties only, so it makes sense to define: A **metric space** is a pair (X, d) where X is a set and

$$d : X \times X \longrightarrow \mathbb{R}$$

is a **metric** (a function that measures distances between elements of X) that satisfies (a), (b) and (c) for all elements $x, y, z \in X$.

For every $\varepsilon > 0$ and $x \in X$ we denote

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

and call $B_\varepsilon(x)$ the (open) **ε-ball** with center x.

A set $U \subseteq X$ is **open** if

$$\forall x \in U, \ \exists \varepsilon > 0 \ : \ B_\varepsilon(x) \subseteq U.$$

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

Example.

- Every open ball $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ is open.
- Every closed ball $\overline{B}_\varepsilon(x) = \{y \in X \mid d(x,y) \le \varepsilon\}$ is closed.

$$U \textbf{ is open} \iff \forall x \in U,\ \exists \varepsilon > 0 \ : \ B_\varepsilon(x) \subseteq U.$$

Proposition. Let (X, d) be a metric space. Then

(i) $\emptyset$ and X are open,

(ii) arbitrary unions of open sets are open, and

(iii) intersections of <u>finitely</u> many open sets are open.

Proof.

Corollary. A set is open if and only if it is a union of open balls.

Proof.

Example. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. This the the **usual metric** of real numbers.

- Open balls:

- Open sets:

- Closed intervals $[a, b]$ are examples of closed sets.

- Set $\mathbb{Q}$ of rational numbers is not open, not closed

- Clopen sets: $\emptyset$ and $\mathbb{R}$.

(i) $\emptyset$ **and** X **are open,**

(ii) **arbitrary unions of open sets are open, and**

(iii) **intersections of finitely many open sets are open.**

Many properties of metric spaces can be proved using properties (i), (ii) and (iii) only.

Further abstraction: A pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a family of subsets of X is a **topological space**, family $\mathcal{T}$ is called a **topology** on X, and sets in $\mathcal{T}$ are called **open** if axioms (i), (ii) and (iii) are satisfied.

Thus the family of open sets of a metric space (X, d) forms a topology on X. It is called a **metric topology**. There are also topologies that are not metrizable, i.e., not defined by any metric.

Example. For any X, let $\mathcal{T}$ contain all subsets of X. Then $\mathcal{T}$ is a topology, the **discrete topology** of X.

The discrete topology is metrizable as it is defined by the discrete metric

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

This metric satisfies the (strong) triangular inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

All singleton sets $\{x\}$ are open balls.

Example. For any set X let $\mathcal{T} = \{X, \emptyset\}$. Then $\mathcal{T}$ is a topology, the **trivial topology** of X.

If $|X| \geq 2$ then $\mathcal{T}$ is not defined by any metric:

Consistently with metric spaces:

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

By de Morgan's laws closed sets behave dually to open sets:

Proposition. Let $(X, \mathcal{T})$ be a topological space.

(i) $\emptyset$ and X are closed,

(ii) arbitrary intersections of closed sets are closed, and

(iii) unions of <u>finitely</u> many closed sets are closed.

Further terminology: Let $(X, \mathcal{T})$ be a top. space.

- $x \in X$ is **isolated** if $\{x\}$ is open. In the metric case:

- A space is **perfect** if it has no isolated points.

- Let $A \subseteq X$. The **closure** of A is

$$\overline{A} = \bigcap_{\substack{F closed \\ A \subseteq F}} F.$$

It is the smallest closed set that contains A:

$$F \text{ closed}, A \subseteq F \implies \overline{A} \subseteq F.$$

- Set $A \subseteq X$ is **dense** if $\overline{A} = X$.

Denseness of a set is proved by showing that it has a non-empty intersection with every non-empty open set:

Lemma. A set $A \subseteq X$ is dense if and only if for every open $U \neq \emptyset$ it holds that $A \cap U \neq \emptyset$.

Proof.

• Dual to closure: The **interior** of A is

$$A^\circ = \bigcup_{\substack{V\, open\\ V \subseteq A}} V.$$

It is the largest open subset of A:

$$V \text{ open}, V \subseteq A \implies V \subseteq A^\circ.$$

• A set A is a **neighborhood** of point x if $x \in A^\circ$. Equivalently: there exists open U such that $x \in U \subseteq A$.

Example. Consider $\mathbb{R}$ and the **usual topology**.

• Every open ball contains infinitely many points so there are no isolated points. The space is perfect.

• The closure of $\mathbb{Q}$ is $\mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$. The interior of $\mathbb{Q}$ is the empty set.

• The closure of $(0, 1)$ is $[0, 1]$.

• $\mathbb{Z}$ is closed, so it is its own closure.

Example. The **discrete topology** is far from perfect because every point is isolated.

Let $A \subseteq X$ and let d be a metric on X. Then d restricted to $A \times A$ is the **induced metric** on A.

Let $A \subseteq X$ and let $\mathcal{T}$ be a topology on X. Then

$$\{V \cap A \mid V \in \mathcal{T}\}$$

is a topology on A, the **induced topology**.

Let $\mathcal{T}$ be the metric topology defined by d on X. The topology that $\mathcal{T}$ induces on A is the same as the metric topology defined by the induced metric on A.

Always, when considering a subset of a topological (or metric) space, the default is that we assume the induced topology (metric) on A.

Example. The metric induced by the usual metric of $\mathbb{R}$ on subset $\mathbb{Z}$ is

$$d(n, m) = |n - m| \text{ for all } n, m \in \mathbb{Z}.$$

Then every singleton set $\{n\}$ is an open ball, and hence the induced topology on $\mathbb{Z}$ is the discrete topology. The discrete metric

$$d(n, m) = \begin{cases} 1, & \text{if } n \neq m, \\ 0, & \text{if } n = m \end{cases}$$

defines the same topology.

Convergence of sequences

A topological space $(X, \mathcal{T})$ is **Hausdorff** if for every $x \neq y$ there are open U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. In other words, any two distinct points have non-intersecting neighborhoods:

Example. Every metric space is Hausdorff: For $x \neq y$ choose

$$\varepsilon = d(x, y)/2$$

and use

$$\begin{aligned} U_x &= B_\varepsilon(x), \\ U_y &= B_\varepsilon(y). \end{aligned}$$

$$\text{Metric} \implies \text{Hausdorff} \implies \text{Topology}$$

The trivial topology $\{\emptyset, X\}$ is not Hausdorff if $|X| \geq 2$.

In a Hausdorff space the singleton sets $\{x\}$ are closed: For every $y \neq x$ there exists an open set V_y such that $x \notin V_y$. The complement of $\{x\}$ is

$$\bigcup_{y \neq x} V_y,$$

thus open as a union of open sets.

A sequence $x_1, x_2, \dots$ **converges** to x if for every open neighborhood U of x there is $n \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq n$.

In the metric setting: For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_i, x) < \varepsilon$ for all $i \geq n$.

Example. Under the trivial topology $\{\emptyset, X\}$ every sequence converges to every point!

Proposition. In a Hausdorff topology every converging sequence converges to a unique point.

Proof.

We denote the unique limit by $\lim_{i\to\infty} x_i$.

Base of a topology

A family $\mathcal{B} \subseteq \mathcal{T}$ is a **base** of topology $\mathcal{T}$ iff every open set is a union of some members of $\mathcal{B}$.

Example. In a metric space (X, d) open sets are precisely unions of open balls. Thus the family

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$$

of all open balls is a base.

Proposition. A family $\mathcal{B} \subseteq \mathcal{T}$ is a base of topology $\mathcal{T}$ if and only if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \ : \ x \in B \subseteq U.$$

Proof.

Compactness

Let $\mathcal{T}$ be a topology on X, and let $A \subseteq X$.

A family $\mathcal{U} \subseteq \mathcal{T}$ is called an **open cover** of A if

$$A \subseteq \bigcup_{V \in \mathcal{U}} V.$$

A subfamily $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathcal{U}$ is called a **subcover** if it is also a cover of A.

Set $A \subseteq X$ is called **compact** if every open cover of A has a finite subcover of A. The topology is called compact if the whole space X is compact.

In other words: a topology is compact iff every family of open sets whose union is X has a finite subfamily whose union is X.

Example. In the usual topology of $\mathbb{R}$

$$A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$$

is compact:

On the other hand,

$$B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$$

is not compact:

Compactness of X could as well be defined using a dual concept:

Proposition. Topology of X is compact if and only if every family of closed sets whose intersection is empty has a finite subfamily whose intersection is empty.

Corollary. Let

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

be an infinite chain of closed sets in a compact space X. If $F_i \neq \emptyset$ for all i then

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

Compactness in metric spaces is equivalent to sequential compactness. We only need one direction of the equivalence:

Proposition. Let X be a compact metric space. Every sequence of elements of X has a converging subsequence.

(A **subsequence** of a sequence $x_1, x_2, \ldots$ is a sequence $x_{i_1}, x_{i_2}, \ldots$ for some $i_1 < i_2 < \ldots$.)

Proof.

In compact metric spaces compact sets are exactly the closed sets:

Proposition A. Let X be a **compact** topological space. For $A \subseteq X$

$$A \text{ closed} \implies A \text{ compact}.$$

Proposition B. Let X be a **Hausdorff** topological space. For $A \subseteq X$

$$A \text{ compact} \implies A \text{ closed}.$$

Proofs.

Countability

Proposition. A compact metric space has a countable base and a countable dense set of points.

Proof.

Baire property

Set $A \subseteq X$ is **residual** if it is the intersection of countably many dense open sets. A topological space X is a **Baire space** if every residual set is dense.

That is: in a Baire space, if $U_1, U_2, \ldots$ are open sets such that $\overline{U}_i = X$ for all i then also $\overline{A} = X$ where

$$A = \bigcap_{i=1}^{\infty} U_i.$$

Example. Set $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is **not** a Baire space:

For every $q \in \mathbb{Q}$ the set $\mathbb{Q} \setminus \{q\}$ is open and dense, but the (countable) intersection

$$\bigcap_{q \in \mathbb{Q}} \mathbb{Q} \setminus \{q\}$$

is empty.

Baire property

Set $A \subseteq X$ is **residual** if it is the intersection of countably many dense open sets. A topological space X is a **Baire space** if every residual set is dense.

That is: in a Baire space, if $U_1, U_2, \ldots$ are open sets such that $\overline{U}_i = X$ for all i then also $\overline{A} = X$ where

$$A = \bigcap_{i=1}^{\infty} U_i.$$

Proposition. Every compact metric space is a Baire space.

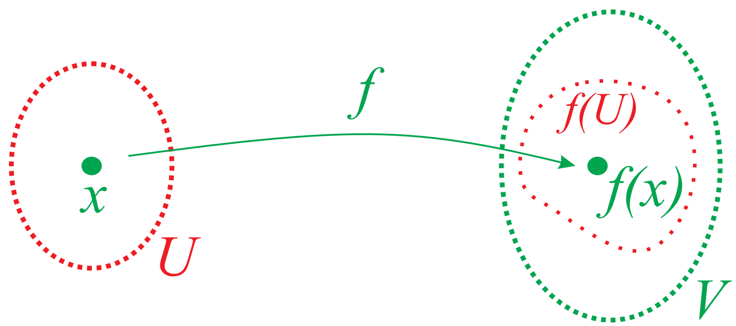
Proof.

Continuity

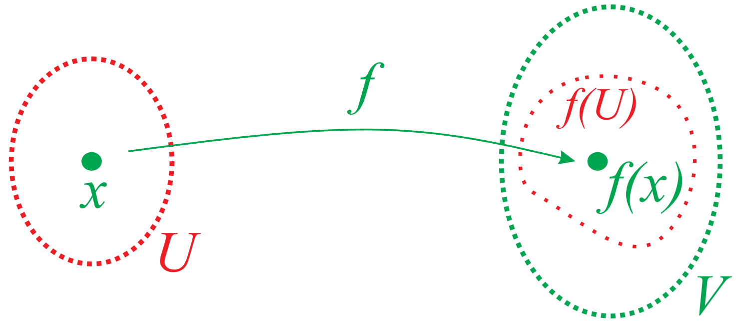
Let X, Y be topological spaces. Function $f : X \longrightarrow Y$ is **continuous at point** $x \in X$ if

$$V \subseteq Y \text{ open}, \ f(x) \in V \\ \implies \exists \text{ open } U \subseteq X : x \in U \text{ and } f(U) \subseteq V.$$

(For every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subseteq V$.



Function $f : X \longrightarrow Y$ is **continuous** if it is continuous at every $x \in X$.



Examples.

• If X has the discrete topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = \{x\}$.)

• If Y has the trivial topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = X$: works because $V = Y$.)

• A constant function ($\forall x \in X : f(x) = a$ for some fixed $a \in Y$) is continuous. (Choose $U = X$.)

• If X has the trivial topology and Y the discrete topology then constant functions are the only continuous functions:

Proposition. The following conditions are equivalent:

(i) Function $f : X \longrightarrow Y$ is continuous,

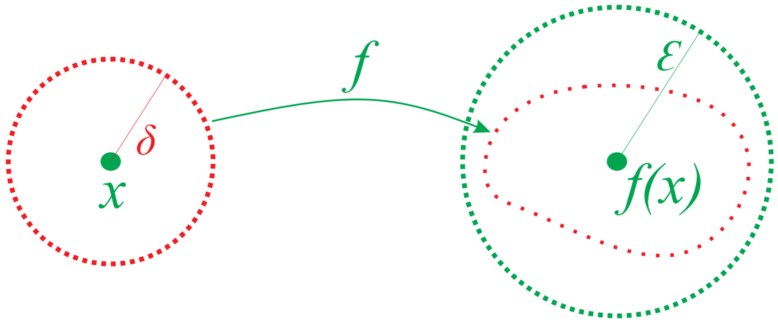
(ii) pre-image $f^{-1}(V)$ is open for each open $V \subseteq Y$,

(iii) pre-image $f^{-1}(F)$ is closed for each closed $F \subseteq Y$.

Proof.

In the metric case: f is continuous if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists \delta > 0\ :\ f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$



Number δ may depend on point x. If δ can be chosen independently of x then function f is **uniformly continuous**:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X\ :\ f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$

Every uniformly continuous function is continuous. In the compact cases also the converse holds:

Proposition. Let X and Y be compact metric spaces. Function $f : X \longrightarrow Y$ is continuous if and only if it is uniformly continuous.

If there exists a positive constant r such that $\delta = r \cdot \varepsilon$ works for all x and ε, then f is **Lipschitz continuous**.

It is an **isometry** if distances remain unchanged:

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Clearly:

$$\begin{gathered} f \text{ isometry} \\ \Downarrow \\ f \text{ Lipschitz continuous} \\ \Downarrow \\ f \text{ uniformly continuous} \\ \Downarrow \\ f \text{ continuous} \end{gathered}$$

Note: Continuity is defined for all topological spaces; the other three concepts only for metric spaces.

Proposition. Let X be a metric space and Y a topological space. Then $f : X \longrightarrow Y$ is continuous if and only if for every converging sequence $x_1, x_2, \ldots$ the sequence $f(x_1), f(x_2), \ldots$ converges and

$$\lim_{i\to\infty} f(x_i) = f(\lim_{i\to\infty} x_i).$$

Proof.

Proposition. Let $f : X \longrightarrow Y$ be continuous. For every compact A the set $f(A)$ is compact.

Proof.

Proposition. Let $f : X \longrightarrow Y$ be a continuous bijection where X is a compact and Y is a Hausdorff topological space. Then the inverse function $f^{-1} : Y \longrightarrow X$ is also continuous.

Proof.

If $f : X \longrightarrow Y$ is a bijection and both f and f^{-1} are continuous then f is a **homeomorphism** and spaces X and Y are **homeomorphic**. This is the "isomorphism" of topological structures.

Corollary. Continuous bijection between compact metric spaces is a homeomorphism.

The Cantor space

Let A be a non-empty finite set, the **alphabet**, whose elements are **letters**.

For any (countable, infinite) set M, the set

$$A^M$$

consists of all assignments $c : M \longrightarrow A$ of letters to elements in M. These are called **configurations**, or simply points of space A^M.

Configurations are “colorings” of M by letters. We often denote $c(m)$ by c_m.

Example. If $M = \mathbb{N}$ then a configuration c is a (one-way) infinite **word**

$$c_0 c_1 c_2 \ldots$$

where $c_i = c(i) \in A$. For example,

$$\begin{array}{l} 001010011101000110\ldots \\ 101010101010\ldots \\ 000000000000\ldots \end{array}$$

are elements of $\{0, 1\}^{\mathbb{N}}$.

Example. If $M = \mathbb{Z}$ then a configuration c is a **bi-infinite word**

$$\ldots c_{-2}c_{-1}.c_0c_1c_2\ldots$$

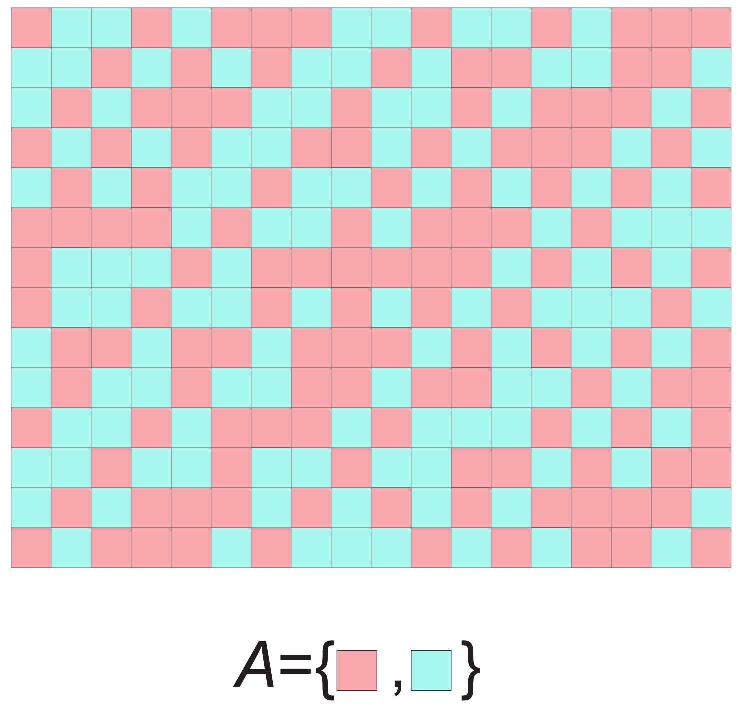
(One may write the dot "." before the position 0 to indicate the origin.)

For example,

$$\ldots 00101001.1101000110 \ldots$$
$$\ldots 10101.0101010 \ldots$$
$$\ldots 00000.0000000 \ldots$$

are elements of $\{0, 1\}^{\mathbb{Z}}$.

Example. If $M = \mathbb{Z}^d$ then a configuration c is a coloring of the d-dimensional grid:



We mostly consider $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$, the sets of infinite words. We then also need the concept of finite words.

Finite words.

A finite **word** over alphabet A is a finite sequence of letters. The **length** of a word w is denoted $|w|$.

$$01101, 010, 0, \varepsilon$$

are words over the alphabet $A = \{0, 1\}$. The **empty word** ε has length zero.

The letters of word w are indexed 0,1,2,etc. and we denote its i's letter by w_i, so

$$w = w_0 w_1 \dots w_{n-1}$$

for w of length n.

A^n is the set of words of length n.

A^* is the set of all finite words. So

$$A^* = A^0 \cup A^1 \cup A^2 \cup \dots$$

A **language** is a set $L \subseteq A^*$ of words over a fixed alphabet. For example

$$\begin{array}{l} \{0, 101, 1101\} \\ \{\varepsilon, 0, 00, 000, 0000, \dots\} \\ \emptyset \\ \{\varepsilon\} \end{array}$$

are languages over the alphabet $\{0, 1\}$. The second language is infinite, the others are finite.

The **concatenation** of words u and v is uv. Its length is $|uv| = |u| + |v|$. The n-fold repetition of word w $ww\ldots w$ is denoted by w^n. The notations may be combined in a single word, and we use parentheses as needed:

$010(10)^3001^401 =$

A **prefix** is any sequence of leading symbols of a word.

A **suffix** is any sequence of trailing symbols of a word.

A **subword** is any sequence of consecutive symbols that appears in a word.

For example, word $w = 010$ has the following prefixes, suffixes and subwords:

For a word u of length n we denote

$$u_{[m,k]} = u_m u_{m+1} \dots u_k$$

(where $0 \leq m \leq k < n$) for the subword from position m to position k.

$$u_{[m,k)} = u_m u_{m+1} \dots u_{k-1}$$

is the subword up to (but not including) position k. Similarly we define $u_{(m,k]}$ and $u_{(m,k)}$.

$001010001_{[2,4]} =$

$001010001_{[2,4)} =$

$001010001_{(2,4)} =$

We use similar notations also on infinite words (=configurations) $c \in A^{\mathbb{N}}$ and $c \in A^{\mathbb{Z}}$.

We also have **infinite repetitions:** For a non-empty finite word u we denote

- by u^∞ the one-way infinite periodic word
$$uuu\ldots$$
- by ${}^\infty u$ the analogous left-infinite repetition
$$\ldots uuu$$
which is an element in $A^{-\mathbb{N}}$
- by ${}^\infty u^\infty$ the bi-infinite periodic word
$$\ldots uu.uu\ldots$$

$$(01)^4(10)^{\infty} =$$

$${}^{\infty}(01)1.110^{\infty} =$$

$$(01)^4(10)^{\infty}\ {}_{[6,8]} =$$

$${}^{\infty}(01)1.110^{\infty}\ {}_{[-2,2]} =$$

Next we define a **metric** on the set $X = A^{\mathbb{N}}$, and prove that the space we obtain is **compact**.

For $x, y \in A^{\mathbb{N}}$ we define

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-\min\{k \mid x_k \neq y_k\}}, & \text{if } x \neq y. \end{cases}$$

The metric considers two configurations to be close to each other if they have a long common prefix (=one needs to look far to see the first difference).

$0101010101001010 1\ldots$ and
$0101011101010101 0\ldots$

are closer to each other than

$010000000000000\ldots$ and
$000000000000000\ldots$

$$d(x,y) = \begin{cases} 0, & \textbf{if } x = y\textbf{,} \\ 2^{-\min\{k \;\mid\; x_k \neq y_k\}}, & \textbf{if } x \neq y\textbf{.} \end{cases}$$

Proposition. Function $d : A^{\mathbb{N}} \times A^{\mathbb{N}} \longrightarrow \mathbb{R}$ is a metric.

Proof.

$$d(x,y) = \begin{cases} 0, & \textbf{if } x = y\textbf{,} \\ 2^{-\min\{k \mid x_k \neq y_k\}}, & \textbf{if } x \neq y\textbf{.} \end{cases}$$

For any $r \in \mathbb{R}$ and $x, y \in A^{\mathbb{N}}$ we have

$$d(x,y) < 2^{-r} \iff y_k = x_k \text{ for all } k \leq r.$$

The open ball of radius $\varepsilon = 2^{-r}$ centered at $x \in A^{\mathbb{N}}$ is then

$$B_\varepsilon(x) = \{y \in A^{\mathbb{N}} \mid y_k = x_k \text{ for all } k \leq r\}.$$

Thus open balls are precisely sets defined by finite prefixes $u \in A^*$ as follows:

$$\{y \in A^{\mathbb{N}} \mid y \text{ has prefix } u\ \}$$

$$u * * * * \ldots$$

More generally, for any finite domain $D \subseteq \mathbb{N}$ and **finite pattern** $w \in A^D$ we define the **cylinder**

$$[w] = \{x \in A^{\mathbb{N}} \mid x_{|D} = w\}$$

of configurations that contain pattern w in domain D. We call D the **shape** of the pattern and the cylinder.

For example, configurations

$$* * 0 * 10 * * * * \dots$$

are in a cylinder $[w]$ of shape $D = \{2, 4, 5\}$.

Open balls are precisely cylinders of shapes $\{0, 1, \dots, r\}$ for some r.

If $D \subseteq E$ are two finite domains then every cylinder $[w]$ of shape D is a finite union of cylinders of shape E:

$$[w] =$$

Any finite $D \subseteq \mathbb{N}$ is a subset of $E = \{0, 1, \ldots, r\}$ for a big enough r
$\Longrightarrow$ Every cylinder is a finite union of open balls, and hence **open**.

Cylinders of shape D form a finite partitioning of $A^{\mathbb{N}}$
$\Longrightarrow$ every cylinder is also **closed**.

Conclusion: Cylinders are a countable **clopen** base of the topology.

Topological space $A^{\mathbb{N}}$ is also **perfect**: it has no isolated points because cylinders are infinite sets.

Theorem. The metric space $(A^{\mathbb{N}}, d)$ is compact. Thus it is a perfect and compact metric space that has a countable clopen base.

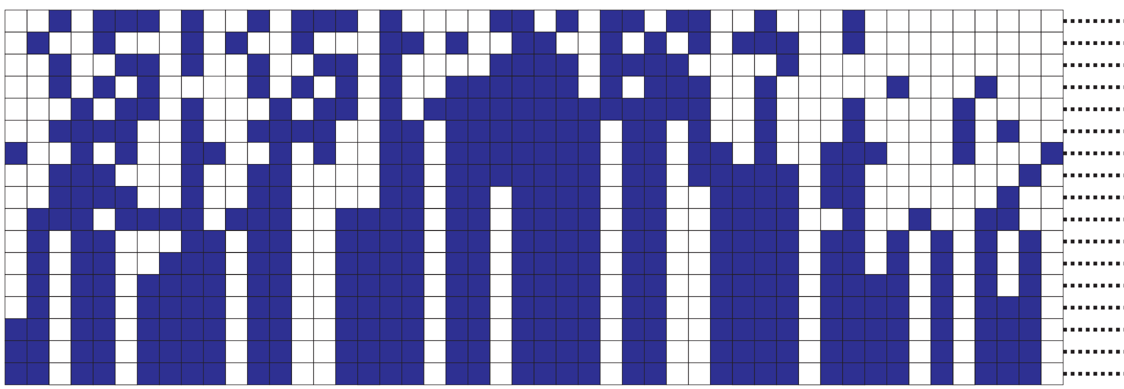
Proof.

By compactness, every sequence of configuration has a converging subsequence. What does convergence mean in our metric ?

Lemma. A sequence $x^{(1)}, x^{(2)}, \ldots$ of configurations of $A^{\mathbb{N}}$ converges to $x \in A^{\mathbb{N}}$ if and only if

$$\forall k \in \mathbb{N}, \exists n \in \mathbb{N}, \forall i > n \ : \ x_k^{(i)} = x_k.$$

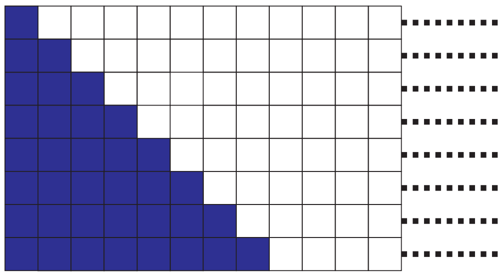
(For every cell $k \in \mathbb{N}$, the symbol in k becomes fixed after a finite initial part of the sequence.)



Proof.

Examples.

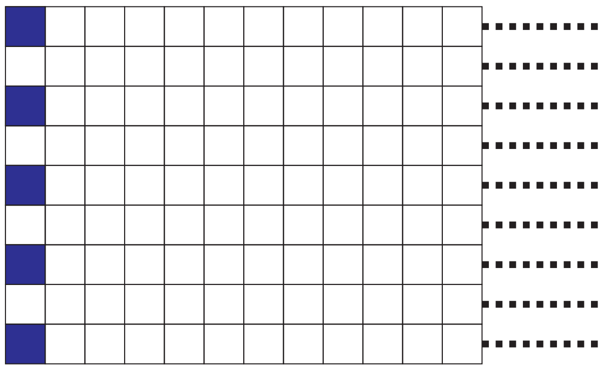
$x^{(n)} = 1^n 0^\infty$



The sequence $x^{(1)}, x^{(2)}, \ldots$ converges to

Examples.

$y^{(n)} = a0^{\infty}$ where $a = 0$ for even n and $a = 1$ for odd n



Does this sequence converge?

Analogous metric topology exists on A^M for any countable infinite M.

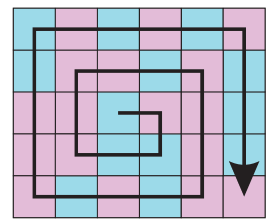
Cylinders of A^M are defined by **patterns** $w \in A^D$ of finite **shapes** $D \subseteq M$:

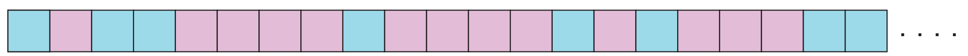
$$[w] = \{x \in A^M \mid x_{|D} = w\}.$$

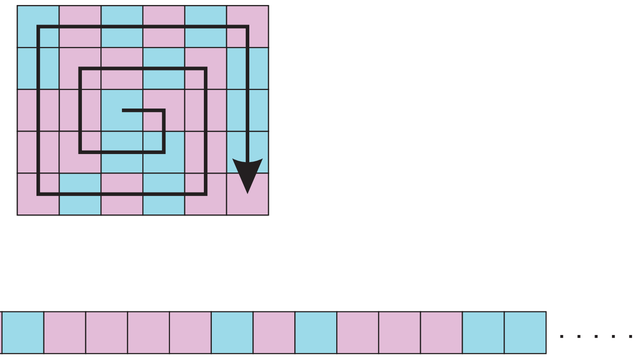
A bijection $\alpha : \mathbb{N} \longrightarrow M$ (=an enumeration of elements of M) defines a bijection $\hat{\alpha} : A^M \longrightarrow A^{\mathbb{N}}$

$$\hat{\alpha}(x) = x_{\alpha(0)} x_{\alpha(1)} x_{\alpha(2)} \ldots$$

For example, a "spiral" enumeration of $\mathbb{Z}^2$ maps a configuration of $A^{\mathbb{Z}^2}$ into $A^{\mathbb{N}}$ as follows:







Define metric on A^M so that $\hat{\alpha}$ is an isometry: the distance of $x, y \in A^M$ is $d(\hat{\alpha}(x), \hat{\alpha}(y))$.

Function $\hat{\alpha}$ maps cylinders to cylinders so the topology on A^M has a clopen base consisting of the cylinders of A^M.

Remark: The concept of a cylinder does not depend on the choice of α: different choices of α define the same topology.

Convergence of sequences of elements of A^M is as in $A^{\mathbb{N}}$: Sequence $x^{(1)}, x^{(2)}, \ldots$ converges to x if for each $m \in M$ we have $x_m^{(i)} = x_m$ for all large enough i.

So we get a compact metrizable topology on $A^{\mathbb{Z}}$, the set of two-way infinite configurations, on multidimensional configuration spaces $A^{\mathbb{Z}^d}$, etc.

All these are homeomorphic to $A^{\mathbb{N}}$. Thus each A^M is a perfect and compact metric space that has a countable clopen base.

It turns out that these properties uniquely characterize the space (up to homeomorphism).

Theorem. Every non-empty perfect and compact Hausdorff space that has a countable clopen base is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

We call such a space **the Cantor space**.

Proof.

So we call a topological space X **the Cantor space** iff it has all these properties:

- $X \neq \emptyset$,
- X is perfect (=no isolated points),
- X is compact,
- X has the Hausdorff property,
- X has a countable clopen base.

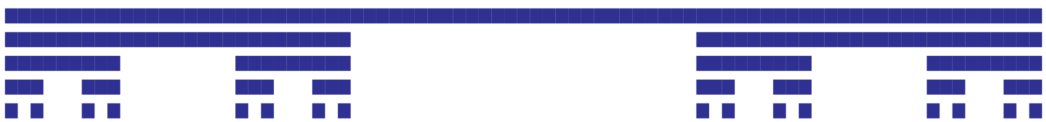
By the theorem such spaces are all homeomorphic with each other, i.e., topologically identical.

A topological space that does not have one or more of these properties is **not** homeomorphic to the Cantor space (because the given properties are all preserved by homeomorphisms.)

Example. The **Cantor set** is the subset of $[0,1] \subseteq \mathbb{R}$ obtained by repeatedly removing the (open) middle third of intervals:

$$\begin{aligned}
F_0 &= [0,1] \\
F_1 &= F_0 \setminus (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] \\
F_2 &= F_1 \setminus \big((\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})\big) = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]
\end{aligned}$$

The Cantor set is the limit $F = F_0 \cap F_1 \cap F_2 \cap \ldots$, with the usual metric of $\mathbb{R}$. Set F is a non-empty compact set because $F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots$ for non-empty closed F_i and $[0,1]$ is compact.



A real number $x \in [0,1]$ is in F if and only if it has a representation in base 3 only using digits 0 and 2. The bijection $f : \{0,2\}^{\mathbb{N}} \longrightarrow F$

$$f(c) = c_0/3 + c_1/9 + c_2/27 + \cdots = \sum_{i=0}^{\infty} c_i/3^{i+1}$$

(c is mapped to the number that $0.c_0c_1c_2\ldots$ represents in base 3) is continuous, and hence a homeomorphism:

$x, y \in \{0,2\}^{\mathbb{N}}$ are close

$\Longrightarrow$ x, y have a long common prefix

$\Longrightarrow$ $f(x), f(y)$ are close in the usual metric of $\mathbb{R}$.