Topological dynamical systems (discrete time)

A **dynamical system** (X, f) consists of a compact metric space X and a continuous function $f : X \longrightarrow X$.

Set X is the **phase space** and f is the **transformation**. If f is bijective (i.e. a homeomorphism) then the system is **invertible**, and the dynamical system (X, f^{-1}) is the **inverse** system.

- The forward **trajectory** of a point $x \in X$ is the sequence

$$x, f(x), f^2(x), \ldots.$$

- The forward **orbit** of x is the set

$$\mathcal{O}(x) = \{f^n(x) \mid n \in \mathbb{N}\}$$

of points on its trajectory.

(So the trajectory is an element of $X^{\mathbb{N}}$ and the orbit is an element of 2^X.)

- A **two-way trajectory** is any bi-infinite sequence

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

that satisfies $x_{i+1} = f(x_i)$ for all $i \in \mathbb{Z}$, and the corresponding **two-way orbit** is the set

$$\{x_i \mid i \in \mathbb{Z}\}$$

of the trajectory.

If the system is invertible the a two-way trajectory is determined by $x = x_0$:

$$\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots$$

- Point $x \in X$ is **periodic** if $f^n(x) = x$, for some $n > 0$.

- x is **eventually periodic** if $\mathcal{O}(x)$ is finite, that is, if $f^n(x) = f^m(x)$ for some $n \neq m$.

- x is a **fixed point** if $f(x) = x$.

- x is **eventually fixed** if $f^{n+1}(x) = f^n(x)$, for some $n \in \mathbb{N}$.

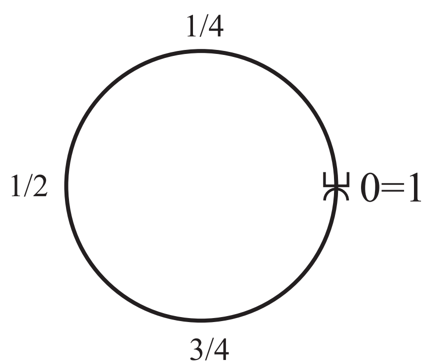
In invertible dynamics eventually periodic points are periodic, and eventually fixed points are fixed points.

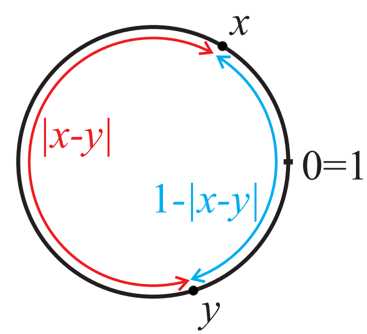
In our examples we we use as phase spaces the Cantor spaces $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$, the unit interval $[0, 1]$ under the usual distance $|x - y|$, and the closed unit circle $\mathbb{T}$.

Example. Let us denote $\mathbb{T} = [0, 1)$, the **circle**, endowed with the metric

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Interpretation: The interval is bent into a ring, gluing together points 0 and 1. Distance d is then the shortest distance between points along the ring.





For $r \in \mathbb{R}$ denote its **integer** and **fractional** parts by

$$\begin{aligned} \lfloor r \rfloor &= \max\{n \in \mathbb{Z} \mid n \leq r\}, \text{ and} \\ \mathrm{frac}(r) &= r - \lfloor r \rfloor. \end{aligned}$$

Then $\mathrm{frac}(r) \in \mathbb{T}$ for all $r \in \mathbb{R}$, and the mapping

$$r \mapsto \mathrm{frac}(r)$$

from $\mathbb{R}$ to $\mathbb{T}$ is (Lipschitz) **continuous** as

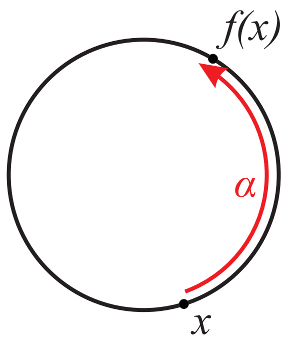
$$d(\mathrm{frac}(x), \mathrm{frac}(y)) \leq |x - y|.$$

The circle space $(\mathbb{T}, d)$ is **compact**: it is the image of the compact set $[0, 1]$ under the continuous map $r \mapsto \mathrm{frac}(r)$.

If $\mathrm{frac}(r) = \mathrm{frac}(s)$ we denote $r = s \pmod 1$.

Example. Let $X = \mathbb{T}$ be the circle. For $\alpha \in \mathbb{R}$ define the **rotation** $\rho_\alpha : \mathbb{T} \longrightarrow \mathbb{T}$ by α as the function

$$x \mapsto \mathrm{frac}(x + \alpha).$$



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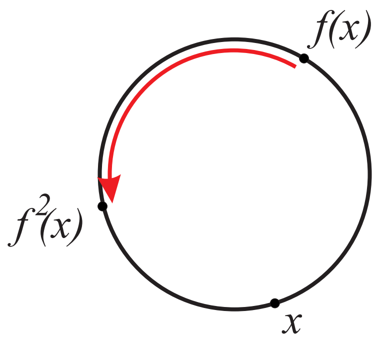

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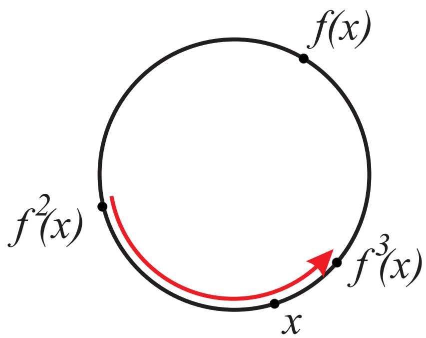

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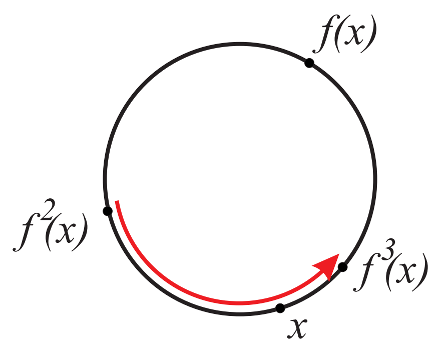


Function ρ_α is an isometry and hence continuous. It is bijective and therefore homeomorphism.

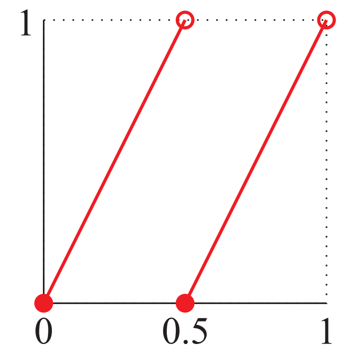
If $\alpha = \frac{m}{n}$ is rational then $f = \rho_\alpha$ has **finite order** because for all $x \in \mathbb{T}$

$f^n(x) =$

so that $f^n = \mathbf{id}_{\mathbb{T}}$. All points are periodic.

If α is irrational then the orbit of every point is **dense**:

Example. The **doubling map** on the circle $\times_2 : \mathbb{T} \longrightarrow \mathbb{T}$ maps $x \mapsto \mathrm{frac}(2x)$:



Function $\times_2$ is (Lipschitz) **continuous** on $\mathbb{T}$, but it is **not injective**:

- Every rational point $x = \frac{m}{n}$ is **eventually periodic**:
- For example, the trajectory of $x = \frac{1}{6}$:
- A point x is **periodic** iff it can be written as $x = \frac{m}{n}$ for an odd n.
- Irrational numbers have **infinite orbits**.

Analogously, for any integer n, we define the multiplication $\times_n : x \mapsto \mathrm{frac}(nx)$ by n.

Example. $X = [0,1]$ under the usual metric,

$$f : x \mapsto x^2.$$

Function f is a homeomorphism of $[0,1]$ so the dynamical system is **invertible**.

Trajectories of all initial points except $x = 1$ converge to limit 0. In the inverse system $([0,1], f^{-1})$ all trajectories except for $x = 0$ converge to 1.

Example. Let $X = A^{\mathbb{N}}$. The one-sided **left shift** $\sigma : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ maps

$$\sigma(x)_k = x_{k+1}$$

for all $x \in A^{\mathbb{N}}$ and all $k \in \mathbb{N}$:

$$x_0\ x_1\ x_2\ x_3 \cdots \mapsto x_1\ x_2\ x_3\ x_4 \ldots$$

Function σ translates infinite words one position to the left, deleting the leftmost symbol. It is **surjective** but **not injective** (if $|A| > 1$): every point has $|A|$ pre-images.

The left shift is **continuous** because the pre-image of a cylinder is a cylinder:

The dynamical system $(A^{\mathbb{N}}, \sigma)$ is called a **one-sided full shift**.

Example. Let $X = A^{\mathbb{Z}}$. The two-sided **left shift** $\sigma : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ maps

$$\sigma(x)_k = x_{k+1}$$

for all $x \in A^{\mathbb{Z}}$ and all $k \in \mathbb{Z}$:

$$\ldots x_{-2}\ x_{-1}\ .\ x_0\ x_1\ x_2 \cdots \mapsto x_{-1}\ x_0\ .\ x_1\ x_2\ x_3 \ldots$$

Function σ translates infinite words one position to the left. It is **bijective**: the inverse function σ^{-1} is the **right-shift**.

Pre-images of cylinders are cylinders $\Longrightarrow$ σ is **continuous**.

Periodic points are precisely ${}^{\infty}u^{\infty}$ for $u \in A^*$, $u \neq \varepsilon$.

The invertible dynamical system $(A^{\mathbb{Z}}, \sigma)$ is a (two-sided) **full shift**.

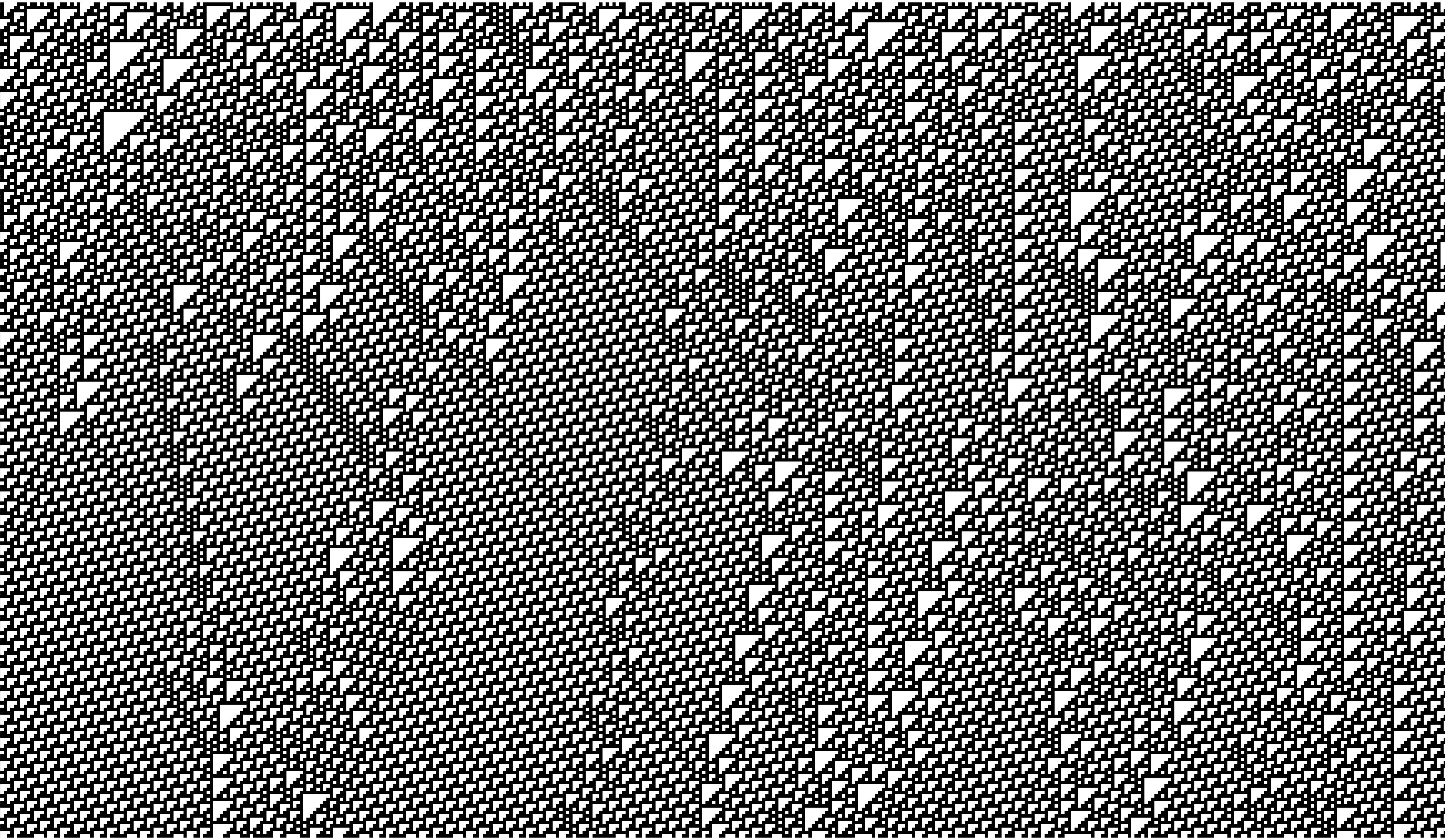
Example. Let $A = \{0, 1\}$ and define **rule 110** as the function $f : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ such that for all $x \in A^{\mathbb{Z}}$ and all $k \in \mathbb{Z}$

$$f(x)_k = \varphi(x_{k-1}, x_k, x_{k+1})$$

where $\varphi : A^3 \longrightarrow A$ is the **local rule** given by the following table:

$$\begin{array}{ccc}
111 & \longrightarrow & 0 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
010 & \longrightarrow & 1 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}$$

(Every cell updates its symbol by applying the rule above on the pattern around the position.)



Rule 110 is **continuous** on $A^{\mathbb{Z}}$:

Let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of configurations that converges to $x \in A^{\mathbb{Z}}$. By convergence

$$\forall k \in \mathbb{Z},\ \exists n \in \mathbb{N},\ \forall i > n \ :\ x^{(i)}_{|\{k-1,k,k+1\}} = x_{|\{k-1,k,k+1\}}.$$

Now $x^{(i)}_{|\{k-1,k,k+1\}} = x_{|\{k-1,k,k+1\}}$ implies that $f(x^{(i)})_k = f(x)_k$, so that

$$\forall k \in \mathbb{Z},\ \exists n \in \mathbb{N},\ \forall i > n \ :\ f(x^{(i)})_k = f(x)_k.$$

This means that $f(x^{(1)}), f(x^{(2)}), \ldots$ converges to $f(x)$.

Thus f is continuous.

Generalizing rule 110. Let $X = A^{\mathbb{Z}}$ for any finite A.

Let $D \subseteq \mathbb{Z}$ be finite, and let

$$\varphi : A^D \longrightarrow A$$

be a function, the **local rule**.

Define $f : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ by: for $x \in A^{\mathbb{Z}}$ and $k \in \mathbb{Z}$

$$f(x)_k = \varphi(\sigma^k(x)_{|D})$$

(where σ is the left shift.)

$\Longrightarrow$ The new symbol in position k is obtained by applying the local rule ϕ on the D-pattern around position k.

Function f is a **cellular automaton**. If f is bijective then f is a **reversible** cellular automaton.

Function f is **continuous** (same proof as for rule 110). Also f **commutes** with the left shift σ:

$$f \circ \sigma = \sigma \circ f.$$

Generalizing further. We can analogously define **one-sided cellular automata** on $A^{\mathbb{N}}$ and **multi-dimensional cellular automata** on $A^{\mathbb{Z}^d}$.

Consider, for example, **Game-of-Life** on $\{0,1\}^{\mathbb{Z}^2}$. Call symbol 0 **dead** and symbol 1 **alive**.

Each cell counts the number of **alive** cells in the eight surrounding cells:

- If the cell is **alive** then it stays alive (survives) iff it has two or three live neighbors. Otherwise it dies of loneliness or overcrowding.
- If the cell is **dead** then it becomes alive iff it has exactly three living neighbors.

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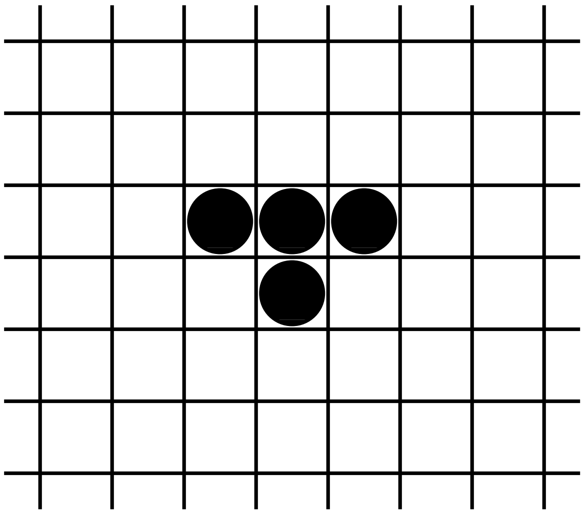

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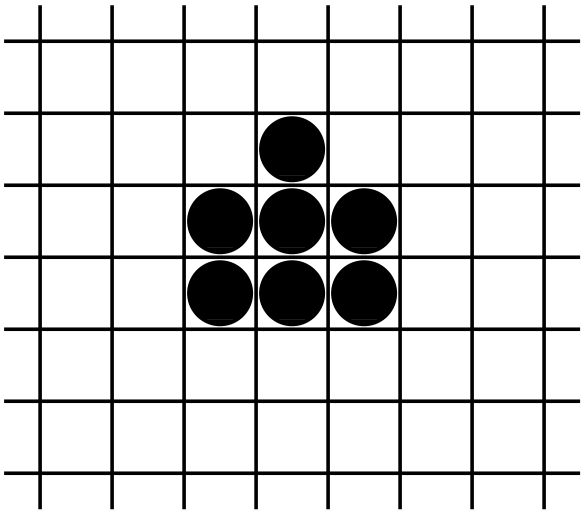

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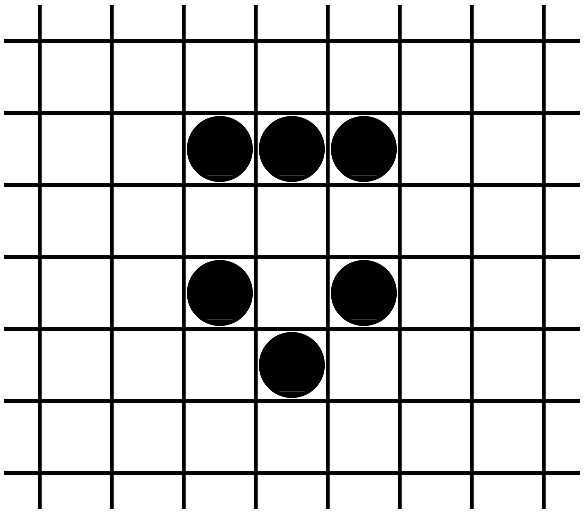

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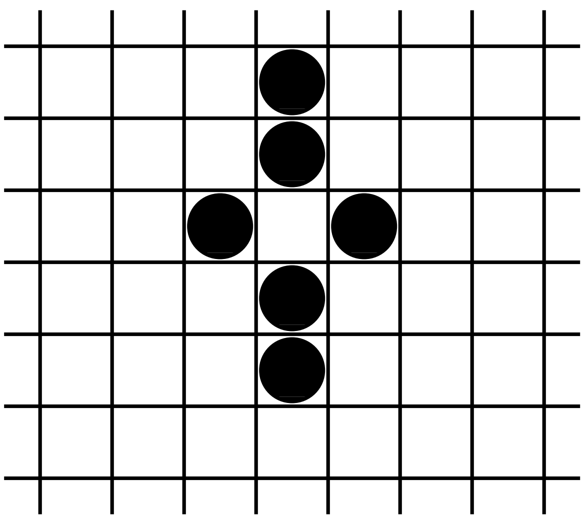

Homomorphisms

Let (X, f) and (Y, g) be dynamical systems. A function $h : X \longrightarrow Y$ is a **homomorphism** if it

- is continuous and
- has the commutation property $h \circ f = g \circ h$.

$$
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[d, "h"'] \\
{} \arrow[r, "g"'] & {}
\end{tikzcd}
$$

In this case $h \circ f^n = g^n \circ h$ for all $n \in \mathbb{N}$:

$$
\begin{tikzcd}
{} \arrow[r, dotted, no head] & {} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[d, "h"'] \arrow[r, dotted, no head] & {} \\
{} \arrow[r, dotted, no head] & {} \arrow[r, "g"'] & {} \arrow[r, "g"'] & {} \arrow[r, "g"'] & {} \arrow[r, "g"'] & {} \arrow[r, dotted, no head] & {}
\end{tikzcd}
$$

The trajectories of f are mapped to trajectories of g.

A homomorphism preserves the two structures we have in dynamical systems: the topology (preserved by continuity) and the transformation (preserved by the commutation property).

- **Factor map** = a **surjective** homomorphism. In this case system (Y, g) is a factor of (X, f).
- **Embedding** = an **injective** homomorphism.
- **Conjugacy** = a **bijective** homomorphism. In this case systems (X, f) and (Y, g) are conjugate.

Conjugacy is thus also a homeomorphism between the metric spaces X and Y. Conjugate systems are "the same": one just has to look into one "through the homeomorphism" h to see the other.

Commutation property implies

$$f^n(x) = x \implies g^n(h(x)) = h(x).$$

So a homomorphism maps a periodic point x to a periodic point $h(x)$ and the shortest period of $h(x)$ has to divide the period of x.

Similarly for eventually periodic points:

$$f^n(x) = f^m(x) \implies g^n(h(x)) = g^m(h(x)).$$

Example: Rotations $(\mathbb{T}, \rho_a)$.

For every α, the system $(\mathbb{T}, \rho_{2\alpha})$ is a factor of $(\mathbb{T}, \rho_\alpha)$.

A factor map is the doubling map $\times_2 : x \mapsto \text{frac}(2x)$. It is continuous and surjective.

The commutation property: For every $x \in \mathbb{T}$:

$(\times_2 \circ \rho_\alpha)(x) =$

Example. There is no embedding of the tripling system $(\mathbb{T}, \times_3)$ into the doubling system $(\mathbb{T}, \times_2)$:

$(\mathbb{T}, \times_2)$ has one fixed point $x = 0$.
$(\mathbb{T}, \times_3)$ has two fixed points 0 and $\frac{1}{2}$.

An injective homomorphism from $(\mathbb{T}, \times_3)$ to $(\mathbb{T}, \times_2)$ would map 0 and $\frac{1}{2}$ to two distinct fixed points of $(\mathbb{T}, \times_2)$.

Example. Let $A = \{0,1\}$ and $B = \{0,1,2\}$ and consider the full shifts $(A^{\mathbb{Z}}, \sigma_A)$ and $(B^{\mathbb{Z}}, \sigma_B)$.

(i) $A^{\mathbb{Z}} \subseteq B^{\mathbb{Z}}$ and the identity $\mathbf{id}_{|A^{\mathbb{Z}}}$ is an embedding of $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$.

(ii) Function h that changes every symbol 2 in all configurations into symbol 1 is a factor map from $B^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$.

(iii) There is no embedding from $(B^{\mathbb{Z}}, \sigma_B)$ to $(A^{\mathbb{Z}}, \sigma_A)$ because the first one has three fixed points and the second one only two.

(iv) There is no factor map from $(A^{\mathbb{Z}}, \sigma_A)$ to $(B^{\mathbb{Z}}, \sigma_B)$. Proof using entropies.

Example. A factor map from full shift $(\{0,1\}^{\mathbb{N}}, \sigma)$ to the doubling system $(\mathbb{T}, \times_2)$.

For $x \in \{0,1\}^{\mathbb{N}}$ denote

$$(x)_2 = \sum_{k=0}^{\infty} x_k 2^{-k-1}$$

for the number in $[0,1]$ with binary expansion $0.x_0x_1x_2x_3\ldots$. We define a factor map $\varphi : \{0,1\}^{\mathbb{N}} \longrightarrow \mathbb{T}$ by

$$x \mapsto \mathrm{frac}((x)_2).$$

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- φ is **continuous**: If $x, y \in \{0,1\}^{\mathbb{N}}$ start with the same prefix of length n then

$$|(x)_2 - (y)_2| \le \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}$$

so that $x \mapsto (x)_2$ is continuous $\mathbb{T} \longrightarrow [0,1]$. We know $\operatorname{frac}(\cdot)$ is continuous $[0,1] \longrightarrow \mathbb{T}$ so φ is a composition of two continuous maps.

- φ is **surjective**: every number has a binary representation.
- **Commutation** property $\varphi \circ \sigma = \times_2 \circ \varphi$: For any $x \in \{0,1\}^{\mathbb{N}}$

$(\varphi \circ \sigma)(x) =$

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Factor map φ is **not injective** because there are numbers with two different binary expansions, one ending $111\ldots$ and one ending $000\ldots$.

$$0.1000\cdots = 0.01111\cdots = \frac{1}{2}.$$

Systems $(\{0,1\}^{\mathbb{N}}, \sigma)$ and $(\mathbb{T}, \times_2)$ are **not conjugate** because $\{0,1\}^{\mathbb{N}}$ and $\mathbb{T}$ are not homeomorphic.

- **Endomorphism** = a homomorphism from a dynamical system to itself.
- **Automorphism** = a conjugacy from a system to itself.

Example. Cellular automata $f : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ are endomorphisms of the full shift $(A^{\mathbb{Z}}, \sigma)$. Reversible cellular automata are automorphisms.

These are the only endomorphisms and automorphisms. (Proof later.)

Example. Circle rotations commute with each other, so ρ_b is an automorphism of $(\mathbb{T}, \rho_a)$ for all $a, b \in \mathbb{R}$. If a is irrational then there are no other automorphisms. (Homework.)

Group, semigroup and monoid.

- A **semigroup** $(G, \star)$ is a set G together with a binary operation $\star : G \times G \longrightarrow G$ that satisfies the axiom of associativity:

$$\forall a, b, c \in G \; : \; (a \star b) \star c = a \star (b \star c).$$

- A **monoid** is a semigroup that also has an identity element:

$$\exists e \in G \; : \; \forall a \in G \; : \; a \star e = e \star a = a.$$

- A **group** is a monoid where every element has an inverse element:

$$\forall a \in G \; : \; \exists b \in G \; : \; a \star b = b \star a = e$$

  where e is the identity element. The inverse element of a (i.e. b above) is usually denoted by a^{-1}.

As with usual multiplication, we usually do not write the operation symbol $\star$ so that associativity, for example, reads

$$(ab)c = a(bc).$$

Recall subgroup/submonoid/subsemigroup of G.

For $A \subseteq G$ we denote $\langle A \rangle$ for the smallest subgroup/ submonoid/subsemigroup of G that contains A.

- G **semigroup**:
  $\langle A \rangle = \{g_1 \star g_2 \star \cdots \star g_n \mid n \geq 1, g_i \in A\}$.

- G **monoid**:
  $\langle A \rangle = \{g_1 \star g_2 \star \cdots \star g_n \mid n \geq 0, g_i \in A\}$.
  (case $n = 0$ gives the identity e.)

- G **group**:
  $\langle A \rangle = \{g_1 \star g_2 \star \cdots \star g_n \mid n \geq 0, g_i \text{ or } g_i^{-1} \in A\}$.

Subset $A \subseteq G$ **generates** G if $\langle A \rangle = G$. We are especially considering finite generating sets A.

Automorphisms of a dynamical system (X, f) form a group under the operation of function composition, the **automorphism group** of (X, f).

Endomorphisms of (X, f) form a monoid, the **endomorphism monoid** of (X, f).

Dynamical systems as monoid actions

In a dynamical system (X, f) the functions f^k are continuous for all times $k \in \mathbb{N}$, and they satisfy $f^{m+k} = f^k \circ f^m$.

We can view the system as a function

$$X \times \mathbb{N} \longrightarrow X$$

that maps $(x, k) \mapsto f^k(x)$.

Time $\mathbb{N}$ is a monoid under addition "+" and property $f^{m+k} = f^k \circ f^m$ links the monoid operation "+" to compositions of f.

We can consider any monoid M as "generalized time": associate to each $m \in M$ a continuous function f^m and require that $f^{mk} = f^k \circ f^m$ holds for all $m, k \in M$.

Let M be a monoid with identity 1_M. A (right) **monoid action** of M on set X is a function

$$f : X \times M \longrightarrow X$$

that satisfies the following conditions (i) and (ii), where for every $m \in M$ we denote $f^m : X \longrightarrow X$ for the function $x \mapsto f(x, m)$:

(i) $f^{1_M} = \mathbf{id}_X$,

(ii) $f^{mk} = f^k \circ f^m$ for all $m, k \in M$.

A **dynamical system over monoid** M is a pair (X, f) where X is a compact metric space and f is a **monoid action** of M on X such that for all $m \in M$ the function f^m is **continuous**.

If M is a group then each f^m is bijective with the inverse function $f^{(m^{-1})}$:

(i) $f^{1_M} = \mathbf{id}_X$**,**

(ii) $f^{mk} = f^k \circ f^m$ **for all** $m, k \in M$**.**

$M = (\mathbb{N}, +)$: This is our standard setting. Function f^1 is the continuous function that we iterate over time:

$$f^k = f^1 \circ f^1 \circ \cdots \circ f^1.$$

$M = (\mathbb{Z}, +)$: Now the monoid is a group and f^1 has to be a bijection (homeomorphism) and f^{-1} is its inverse. Now f^{-k} is an k-fold iteration of f^{-1}.

$M = \langle G \rangle$ **is generated by** $G \subseteq M$: The dynamical system is uniquely determined by functions f^g for $g \in G$:

(i) $f^{1_M} = \mathbf{id}_X$**,**

(ii) $f^{mk} = f^k \circ f^m$ **for all** $m, k \in M$**.**

Example. Let M be a countable monoid (e.g. finitely generated), A a finite set, and consider $X = A^M$.

The **shift action** σ of M on the configuration space A^M is defined by

$$\forall m, k \in M, \forall x \in A^M \ : \ \sigma^m(x)_k = x_{mk}.$$

(When shifting by $m \in M$ the contents of cell mk get moved to cell k.)

Then σ^m is **continuous** for all $m \in M$.

Let us verify the conditions of **monoid actions**:

(i)

(ii)

Why “generalized time” M ?

- Many concepts and proofs work without change for any monoid M in place of $\mathbb{N}$. Even continuous time with monoid $(\mathbb{R}, +)$.
- Especially in symbolic dynamics there is wide interest on multidimensional shift spaces $A^{\mathbb{Z}^d}$ where the shift is a $\mathbb{Z}^d$-action. Also shift spaces A^G for other groups G are studied.
- No need to consider $\mathbb{N}$- and $\mathbb{Z}$-actions (when f bijective) separately.

Trajectory and orbit revisited. Let (X, f) be a dynamical system over monoid M.

- The **trajectory** of point $x \in X$ is $t \in X^M$ defined by $t_m = f^m(x)$ for all $m \in M$.
- The **orbit** of $x \in X$ is the set

$$\mathcal{O}(x) = \{f^m(x) \mid m \in M\}$$

For $M = \mathbb{N}$ these coincide with our earlier concepts of forward trajectory/orbit.

For $M = \mathbb{Z}$ these coincide with our earlier concepts of two-way trajectory/orbit of invertible systems.

Homomorphism revisited. Let (X,f) and (Y,g) be a dynamical systems over the same monoid M.

A **homomorphism** from (X,f) to (Y,g) is a function $h : X \longrightarrow Y$ that

- is **continuous**, and
- satisfies the **commutation property**

$$\forall m \in M \ : \ h \circ f^m = g^m \circ h.$$

For $M = \mathbb{N}$ or $M = \mathbb{Z}$ (and invertible system) this coincides with our earlier concept of a homomorphism.

If $M = \langle A \rangle$ is generated by $A \subseteq M$ then it is enough to verify the commutation property on the generators:

$$\forall m \in A \ : \ h \circ f^m = g^m \circ h.$$

Subsystems

Let (X, f) be a dynamical system over any monoid M.

A subset $Y \subseteq X$ of **invariant** if

$$\forall m \in M \ : \ f^m(Y) \subseteq Y.$$

Example. The orbit $\mathcal{O}(x)$ of any $x \in X$ is invariant.

If $Y \neq \emptyset$ is

- **closed** and
- **invariant**

then

$$(Y, f_{|Y \times M})$$

is a dynamical system, a **subsystem** of (X, f). We also simply call Y a subsystem of (X, f).

Remark: In the classical setting it makes a difference whether the system is considered as an

- $\mathbb{N}$-system (only positive time): forward trajectories of $x \in Y$ have to stay inside Y, or
- or a $\mathbb{Z}$-system (also negative time): bi-infinite orbits need to stay in Y.

Example. Full shifts $(A^{\mathbb{Z}}, \sigma)$ have many subsystems. These are called (two-sided) **subshifts**. For any language $L \subseteq A^*$ the set

$$\{x \in A^{\mathbb{Z}} \mid \text{ no word of } L \text{ is a subword of } x \ \}$$

is a subshift (if non-empty).

Lemma. Let $Y \subseteq X$ be invariant for M-system (X, f) and $Y \neq \emptyset$. Then the closure $\overline{Y}$ is a subsystem of (X, f).

Proof.

The **orbit closure** $\overline{\mathcal{O}(x)}$ of $x \in X$ is the topological closure of the orbit of $x \in X$.

Corollary. The orbit closure $\overline{\mathcal{O}(x)}$ of every $x \in X$ is a subsystem, the smallest subsystem that contains x.

Point $x \in X$ is **transitive** if $\overline{\mathcal{O}(x)} = X$.

Example. Circle rotation $(\mathbb{T}, \rho_\alpha)$. We can view it as a system over $\mathbb{N}$ or $\mathbb{Z}$.

- If α is irrational then all $x \in \mathbb{T}$ are transitive: already the forward orbits are all dense.
- If α is rational then every $x \in \mathbb{T}$ has a finite orbit. Each such finite orbit is a subsystem.

A dynamical system (X, f) is **minimal** if it has no proper subsystems.

Proposition. Dynamical system (X, f) over M is minimal if and only if all $x \in X$ are transitive points.

Proof.

Example. Circle rotations $(\mathbb{T}, \rho_\alpha)$ with irrational α are minimal.

Mixing properties.

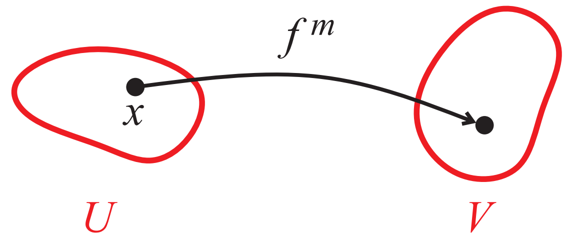
By mixing properties of a dynamical system $g : X \longrightarrow X$ we mean its tendency to mix different parts of its phase space X. Mixingness comes in different variants of various strengths.

First mixing property: Transitivity

Dynamical system (X, f) over monoid M is **transitive** if for all non-empty open $U, V \subseteq X$ there exists $m \in M$ such that

$$\exists x \in U \ : \ f^m(x) \in V$$

$\forall\, U,V$: $\exists m,x$:



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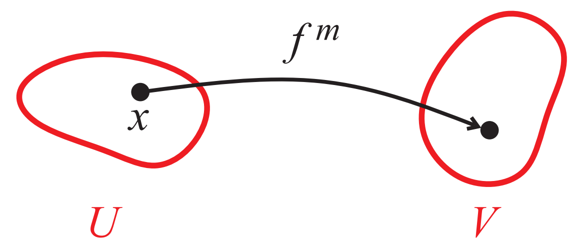
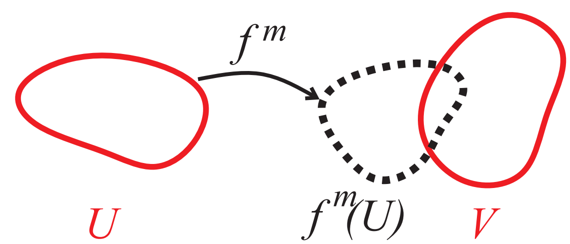


or equivalently

$$f^m(U) \cap V \neq \emptyset$$

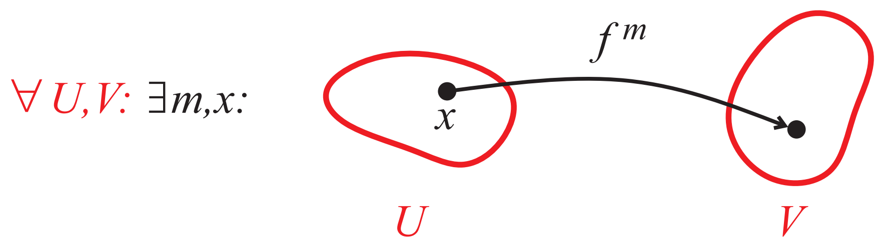
$\forall\, U,V$: $\exists m$:



First mixing property: Transitivity

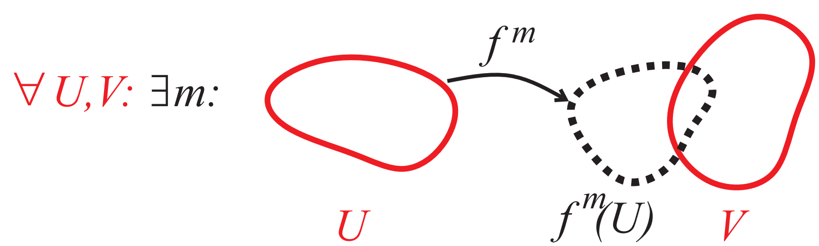
Dynamical system (X, f) over monoid M is **transitive** if for all non-empty open $U, V \subseteq X$ there exists $m \in M$ such that

$$\exists x \in U \ : \ f^m(x) \in V$$



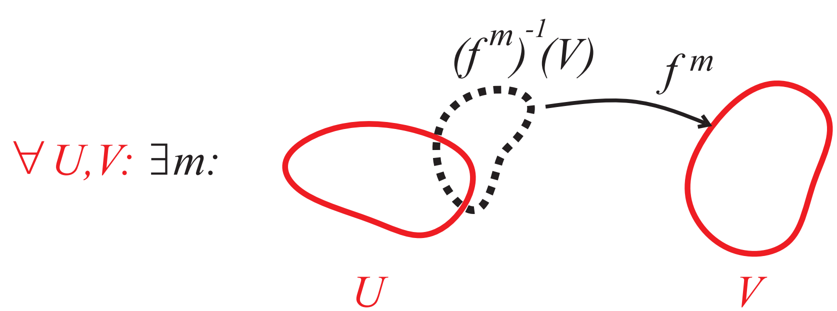
or equivalently

$$f^m(U) \cap V \neq \emptyset$$



or equivalently

$$(f^m)^{-1}(V) \cap U \neq \emptyset$$



Proposition. Let M be a **group**. A dynamical system (X, f) over M is transitive if and only if it has a transitive point. In this case, the set of transitive points is residual.

Proof. Let T be the set of transitive points of the system. We prove three implications:

(X, f) is transitive $\Rightarrow$ T is residual $\Rightarrow$ $T \neq \emptyset$ $\Rightarrow$ (X, f) is transitive

First two hold for any monoid M, the third needs M to be a group.

$$(X, f) \text{ is transitive } \Rightarrow T \text{ is residual } \Rightarrow T \neq \emptyset \Rightarrow (X, f) \text{ is transitive}$$

This holds for any monoid M:

$$(X, f) \text{ is transitive } \Rightarrow T \text{ is residual } \Rightarrow T \neq \emptyset \Rightarrow (X, f) \text{ is transitive}$$

This holds for any monoid M.

Proof: The compact metric space X is a Baire space so residual sets are dense, thus non-empty.

$$(X,f) \text{ is transitive } \Rightarrow T \text{ is residual } \Rightarrow T \neq \emptyset \Rightarrow (X,f) \text{ is transitive}$$

Here M is assumed to be a group:

Example. Let $X = \{1, \frac{1}{2}, \frac{1}{4}, \ldots\} \cup \{0\}$ under the usual metric of $\mathbb{R}$.

Let $f(x) = \frac{x}{2}$.

Then (X, f) is a dynamical system over $\mathbb{N}$.
(X compact, f continuous.)

Point $x = 1$ is a transitive point.

The system is not transitive: Consider $U = \{\frac{1}{2}\}$ and $V = \{1\}$.

Remark: Function f is not surjective.

Proposition. A dynamical system (X, f) over $\mathbb{N}$ is transitive if and only if it has a transitive point and $f^1 : X \longrightarrow X$ is surjective. In this case the set of transitive points is residual.

Proof. Let T be the set of transitive points.

1) (X, f) is transitive $\;\Rightarrow\;$ T is residual $\;\Rightarrow\;$ $T \neq \emptyset$
holds in any monoid, including $\mathbb{N}$.

2) (X, f) is transitive $\;\Rightarrow\;$ f^1 is surjective:

3) $T \neq \emptyset$ and f^1 is surjective $\Rightarrow (X, f)$ is transitive:

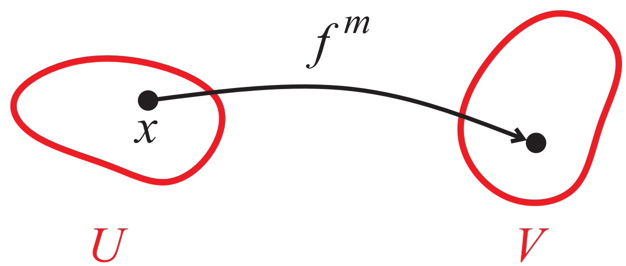
Stronger mixing property: Mixingness

This property we only define in the "classical" setup: Let (X, f) be a dynamical system over the monoid $(\mathbb{N}, +)$. It is **mixing** if for all non-empty open $U, V \subseteq X$

$$f^m(U) \cap V \neq \emptyset$$

holds for all large enough $m \in \mathbb{N}$:

$\forall\, U,V\colon \exists n\colon \forall\, m \geq n\colon \exists x\colon$



A mixing system is transitive.

Also: For any non-empty open $U_1, \ldots, U_s$ and $V_1, \ldots, V_t$ we have that for all large enough m holds that

$$\forall i, j \ : \ f^m(U_i) \cap V_j \neq \emptyset.$$

Example. Assume $|X| \geq 2$. An isometry on X is not mixing:

In particular: a rotation ρ_α of the circle $\mathbb{T}$ is not mixing for any α.

Example.

Full shifts $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ are mixing:

Proposition. Let (X, f) and (Y, g) be dynamical systems over monoid M, and let $h : X \longrightarrow Y$ be a factor map.

(a) If (X, f) is transitive so is (Y, g).

(b) In the case $M = \mathbb{N}$, if (X, f) is mixing so is (Y, g).

Proof.

Example. The doubling map $\times_2$ on the circle $\mathbb{T}$ is mixing: it is the factor of the full shift $\{0,1\}^{\mathbb{N}}$.

Minimality again

Recall: A dynamical system (X, f) is **minimal** if it has no proper subsystems. Equivalently: all $x \in X$ are transitive points.

A minimal system is always transitive: if non-empty U and V are open then by minimality the orbit of every point in U intersects with V.

A minimal system over $M = \mathbb{N}$ does not need to be mixing:

Example. A circle rotation $(\mathbb{T}, \rho_\alpha)$ by irrational α is minimal but not mixing.

Theorem. Every dynamical system (X, f) over any monoid M contains a minimal subsystem.

Recurrence

A periodic point $x \in X$ returns exactly back to its initial value regularly under the iteration of f. This is a very strong form of recurrence.

We consider three weaker types of recurrence:

- **recurrence** (only for systems over the monoid $(\mathbb{N}, +)$),
- **uniform recurrence** (the most important concept of these three; related to minimality),
- **quasi-periodicity** (only for systems over the monoid $(\mathbb{N}, +)$).

Recurrent points

Let (X, f) be a dynamical system over monoid $\mathbb{N}$.

Point $x \in X$ is **recurrent** if it returns back to its every open neighborhood:

$$\forall \text{ open } U \ni x \ : \ \exists k > 0 \ : \ f^k(x) \in U.$$

Lemma. $x \in X$ is recurrent if and only if its forward trajectory has a subsequence that converges to x.

Proof.

Corollary. A recurrent point x returns to each open neighborhood of x infinitely many times.

Uniformly recurrent points

Let (X, f) be a dynamical system over an arbitrary monoid M. Point $x \in X$ is **uniformly recurrent** if for every open neighborhood U of x there exists a finite $R \subseteq M$ such that

$$\forall k \in M \ : \ f^{kr}(x) \in U \text{ for some } r \in R.$$

(At any time k the system can return to U within bounded time, i.e., time belonging to fixed finite R.)

Difference to recurrence on $M = \mathbb{N}$:

- **uniformly recurrent** points keep on returning to their open neighborhoods within **bounded time gaps.** (The bound may be different for different neighborhoods.)
- **recurrent** points return to their open neighborhoods infinitely many times but the time gaps between consecutive returns do not need to be bounded.

Example. One-sided full shift $(A^{\mathbb{N}}, \sigma)$ for $A = \{0, 1\}$.

If $x \in A^{\mathbb{N}}$ is any sequence that contains all $u \in A^*$ as finite subwords then x is recurrent but not uniformly recurrent.

Theorem. The orbit closure $\overline{\mathcal{O}(x)}$ is minimal if and only if x is uniformly recurrent.

Proof.

Corollary. Minimal systems are orbit closures of uniformly recurrent points, and all points of a minimal system are uniformly recurrent. Every dynamical system contains a uniformly recurrent point.

Example. The circle rotation $(\mathbb{T}, \rho_\alpha)$ by irrational α is minimal. So every $x \in \mathbb{T}$ is uniformly recurrent. No point is periodic.

Quasi-periodic points

Let (X, f) be a dynamical system over monoid $M = (\mathbb{N}, +)$ or $M = (\mathbb{Z}, +)$.

A **quasi-periodic** x returns back to each neighborhood periodically:

$$\forall \text{ open } U \ni x \ : \ \exists p \geq 1 \ : \ \forall i \in M \ : \ f^{ip}(x) \in U.$$

Note: the period p may be different for different neighborhoods U.

We clearly have the following implications:

$$\begin{aligned} & x \text{ periodic} \\ & \quad \Longrightarrow x \text{ quasi-periodic} \\ & \quad\quad \Longrightarrow x \text{ uniformly recurrent} \\ & \quad\quad\quad \Longrightarrow x \text{ recurrent} \end{aligned}$$

Example. Rotations $(\mathbb{T}, \rho_a)$ with irrational a. All $x \in \mathbb{T}$ are non-periodic but uniformly recurrent.

There are no quasi-periodic points: for every period candidate p, the set $\{x, \rho_a^p(x), \rho_a^{2p}(x), \ldots\}$ is dense as the orbit of x under the irrational rotation by pa.

Example. Quasi-periodic elements of $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ are precisely the so-called Toeplitz-sequences.

For $M = \mathbb{N}$ or $M = \mathbb{Z}$, a configuration $x \in A^M$ is a **Toeplitz-sequence** if for every $m \in M$ there exists $p > 0$ such that $x_m = x_{m+ip}$ for all $i \in M$.

x **quasi-periodic** $\Longrightarrow$ x **Toeplitz:** Consider visits to the single site cylinder $[x_m]$ at position m.

x **Toeplitz** $\Longrightarrow$ x **quasi-periodic:** every finite pattern in a Toeplitz x is part of an arithmetic progression of identical patterns (the least common multiple of the periods for the individual symbols of the pattern works as the period).

A **concrete example** of a non-periodic Toeplitz-sequence $x \in \{0,1\}^{\mathbb{N}}$: $x = 010001010100\ldots$

x_k **is the parity of the largest** n **such that** 2^n **divides** $k+1$**.**

Sensitivity to initial conditions

Small changes in the initial point of a trajectory may be amplified over time (“butterfly effect”).

- Stable points where such effect does not happen are **equicontinuity points**.
- A system is **equicontinuous** if all points are **equicontinuity points**.
- In contrast, in **sensitive** systems there is a distance $s > 0$ such that arbitrarily close to all points there are points whose trajectories diverge by at least s.
- **Expansivity** is a very strong form of sensitivity where all trajectories diverge from each other by at least s.

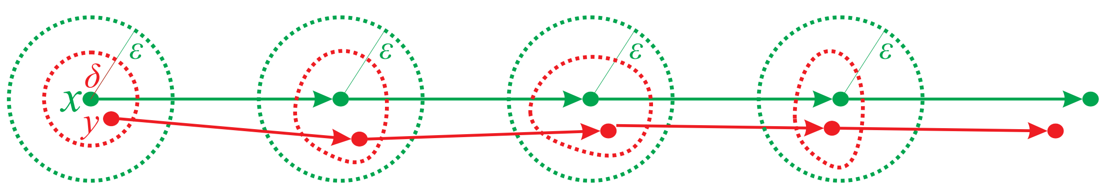
Equicontinuity

Recall the **general term** on metric spaces: a family $\mathcal{S}$ of functions $X \longrightarrow Y$ is called **equicontinuous** at $x \in X$ if all $f \in \mathcal{S}$ are continuous at x using a common parameter value:

$$\forall \varepsilon > 0 \ : \ \exists \delta > 0 \ : \ \forall f \in \mathcal{S} \ : \ f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$

Let (X, f) be a dynamical system over monoid M. The system is **equicontinuous at $x \in X$** if the family $\{f^m \mid m \in M\}$ is equicontinuous at x. In this case we say that x is an **equicontinuity point** of the system.

$$\forall \varepsilon > 0 \ : \ \exists \delta > 0 \ : \ \forall m \in M \ : \ f^m(B_\delta(x)) \subseteq B_\varepsilon(f^m(x)).$$



System (X, f) is **equicontinuous** if all $x \in X$ are equicontinuity points.

System (X, f) is **almost equicontinuous** if the set of equicontinuity points is a residual set.

Example. Any isometry is equicontinuous. Thus the circle rotation $(\mathbb{T}, \rho_a)$ is equicontinuous.

Proposition. Let (X, f) be a transitive dynamical system over a monoid M. Every equicontinuity point is a transitive point.

Proof.

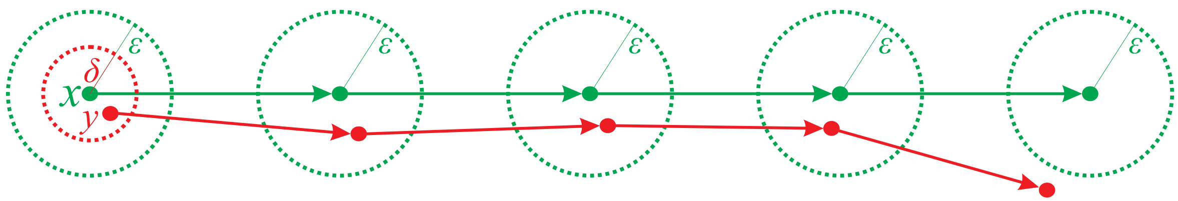
Example. In the circle rotation $(\mathbb{T}, \rho_a)$ all $x \in \mathbb{T}$ are equicontinuity points.

- If α is irrational then the system is transitive. All points are transitive.
- If α is rational then the system is not transitive. No point is transitive.

Sensitivity

Dynamical system (X, f) over monoid M is called **sensitive** if there exists $\varepsilon > 0$, the **sensitivity constant**, such that arbitrarily close to any point there is another point and a time when the trajectories of the two points deviate by at least ε:

$$\exists \varepsilon > 0 \ : \ \forall x \in X \ : \ \forall \delta > 0 \ : \ \exists m \in M \ : \ f^m(B_\delta(x)) \not\subseteq B_\varepsilon(f^m(x)).$$



Complementing gives that the system is **not** sensitive iff

$$\forall \varepsilon > 0 \ : \ \exists x \in X \ : \ \exists \delta > 0 \ : \ \forall m \in M \ : \ f^m(B_\delta(x)) \subseteq B_\varepsilon(f^m(x)).$$

Thus a sensitive system has no equicontinuity points. On transitive system also the converse holds (next proposition).

Proposition. Let M be a group or $M = \mathbb{N}$. Let (X, f) be a transitive dynamical system over M and let $E \subseteq X$ be the set of its equicontinuity points. Then exactly one of the two following conditions holds:

(i) The system is sensitive and $E = \emptyset$.

(ii) The system is not sensitive and E is a residual set (and hence non-empty).

Proof.

Proposition. Let M be a group or $M = \mathbb{N}$. Let (X, f) be a transitive dynamical system over M and let $E \subseteq X$ be the set of its equicontinuity points. Then exactly one of the two following conditions holds:

(i) The system is sensitive and $E = \emptyset$.

(ii) The system is not sensitive and E is a residual set (and hence non-empty).

Proof. We define for every $\varepsilon > 0$ the set

$$E_\varepsilon = \{x \in X \mid \exists \delta > 0, \forall m \in M \ : \ f^m(B_\delta(x)) \subseteq B_\varepsilon(f^m(x))\},$$

and conclude that

- $E = \bigcap_{\varepsilon > 0} E_\varepsilon$,
- for $\varepsilon_1 < \varepsilon_2$ holds $E_{\varepsilon_1} \subseteq E_{\varepsilon_2}$.

There are two possibilties:

1° $E_\varepsilon = \emptyset$ for some $\varepsilon > 0$: then $E = \emptyset$ and (X, f) is sensitive with sensitivity constant ε.

2° $E_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$. Then no $\varepsilon > 0$ works as the sensitivity constant so the system is not sensitive. We just need to show that E is a residual set.

$$E_\varepsilon = \{x \in X \mid \exists \delta > 0, \forall m \in M \ : \ f^m(B_\delta(x)) \subseteq B_\varepsilon(f^m(x))\}$$

- $E = \bigcap_{\varepsilon > 0} E_\varepsilon$.
- For $\varepsilon_1 < \varepsilon_2$ holds $E_{\varepsilon_1} \subseteq E_{\varepsilon_2}$.
- Assume that $E_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$. We want to prove that E is residual.

Consider an arbitrary $\varepsilon > 0$:

(a) We show backward invariance (need here that M is a group or $\mathbb{N}$):

$$\forall k \in M \ : \ f^k(x) \in E_\varepsilon \Longrightarrow x \in E_\varepsilon.$$

(b) Next we show that there is an open set U_ε such that $E_\varepsilon \subseteq U_\varepsilon \subseteq E_{2\varepsilon}$

(c) Finally we show that U_ε found in (b) is dense.

It follows that $E = U_1 \cap U_{1/2} \cap U_{1/3} \cap \ldots$ is residual.

Expansivity

A very strong form of sensitivity to initial conditions is expansivity. It requires that there exists $\varepsilon > 0$, the **expansivity constant**, such that the trajectories of any distinct points eventually deviate by at least ε:

$$\begin{aligned} &\exists \varepsilon > 0 \ : \ \forall x, y \in X \ : \\ &\qquad x \neq y \Longrightarrow (\exists m \in M \ : \ d(f^m(y), f^m(x)) \geq \varepsilon). \end{aligned}$$

Remark: If there are no isolated points in X then

$$(X, f) \text{ expansive } \Longrightarrow (X, f) \text{ sensitive.}$$

(If there is an isolated point in X then no system is sensitive!)

Example. The full shift (A^M, σ) over a monoid M is expansive: If $x, y \in A^M$ and $x \neq y$ then $x_m \neq y_m$ for some $m \in M$. Then $\sigma^m(x)$ and $\sigma^m(y)$ differ at cell 1_M.

A subsystem of an expansive system is also expansive. Thus all subshifts are also expansive.

Chaos

We have studied three families of properties of dynamical systems

- **Mixing** properties (transitivity, mixingness, minimality, transitive points),
- **Regularity** properties (periodicity, quasi-periodicity, recurrence, uniform recurrence),
- **Stability** vs. **sensitivity** properties (equicontinuity, sensitivity, equcontinuity points, expansivity).

In a **chaotic** dynamical system there is regular and transitive behavior densely everywhere in the phase space, with sensitivity so that small changes in the initial state may change regular behavior into transitive behavior or vice versa.

System (X, f) (classical sense, over monoid $(\mathbb{N}, +)$) is **Devaney chaotic** if

(1) it is transitive,

(2) it is sensitive,

(3) periodic points are densely in X.

It turns out that property (2) follows from (1) and (3) in infinite systems (homework).