Tilings and Patterns: Homework 2 (22.9.2025)

1. Determine all symmetries of the following subsets of $\mathbb{R}^2$.

   (a) Circle $\{(x, y) \mid x^2 + y^2 = 1\}$,

   (b) Isosceles (but non-equilateral) triangle,

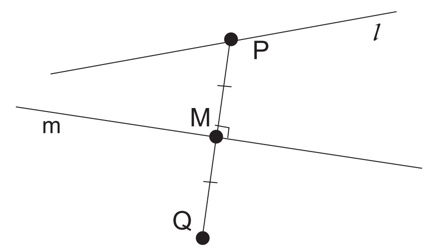
   (c) Equilateral triangle,

   (d) The sine curve $\{(x, \sin x) \mid x \in \mathbb{R}\}$.

2. (a) Prove that two non-trivial rotations with different centers do not commute, that is, demonstrate that $\rho_{A,\Theta}\rho_{B,\Phi} \neq \rho_{B,\Phi}\rho_{A,\Theta}$ when $A \neq B$ and $\Theta, \Phi \neq 0°$. (Hint: See how point A moves.)

   (b) Give a specific example of three non-trivial rotations about three different centers, whose product is the trivial isometry ι.

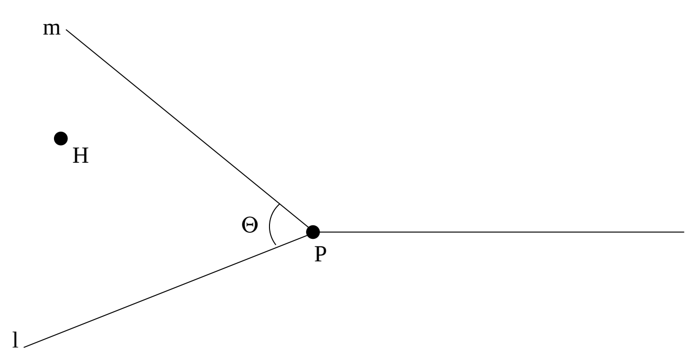
3. Let P and Q be two arbitrary points on the plane. Prove that the isometries that take P to Q are exactly $\sigma_M\sigma_l$ and $\sigma_m\sigma_l$ where l ranges over the lines containing point P, and M and m are the midpoint and the perpendicular bisector of segment PQ, respectively.



4. Given $2n + 1$ points $A_1, A_2, \ldots, A_{2n+1}$, prove that there are unique points $B_1, B_2, \ldots, B_{2n+1}$ such that A_i is the midpoint between points B_i and B_{i+1} for every $i = 1, 2, \ldots, 2n$, and A_{2n+1} is the midpoint between B_{2n+1} and B_1. How do you find points B_i ? Is the claim true if the number of points A_i is even instead of odd ?

5. Given a point P, a circle and a line, demonstrate how one finds (using only a ruler and a compass!) points C and L from the circle and the line, respectively, such that P, C and L are all of equal distance from each other. Show that there are at most four different solutions.

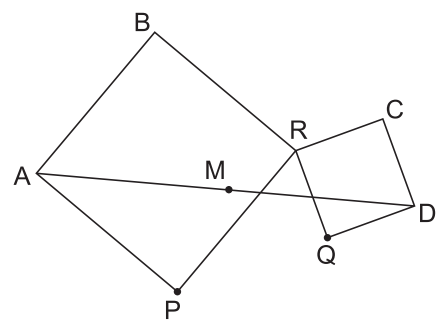
6. House H is between two rivers m and l (both straight lines) that meet at point P at angle $\Theta < 90°$.



   (a) Determine the shortest route that starts in house H, visits both rivers and returns back to H.

(b) Show that the length of the shortest path constructed in (a) is $2r \sin \Theta$, where r is the distance between H and P.

7. Let P and Q be two given points on the plane. Draw two squares $RPAB$ and $RCDQ$ with a common vertex R, and having P and Q as a vertex, respectively. The vertices of the squares are listed in the clockwise orientation, see



Prove that the location of the midpoint M between vertices A and D is uniquely determined by points P and Q and does not depend on the choice of point R. (Hint: Consider $\alpha = \rho_{P,90^\circ} \circ \rho_{Q,90^\circ}$.)

(This problem appears in the following riddle: Someone found in his attic an old description of a pirate who died long time ago. It read as follows: Go to the island X, start at the gallows, go to the elm tree and count the steps. Then turn left by 90°, go the same number of steps and mark the point. Again, go to the gallows and go from there to the fig tree counting steps. Turn 90° right and proceed the same number of steps. Mark the point. A treasure is buried midway between the two marked points. The fellow traveled to the island and found the fig and the elm trees, but could not locate the gallows. Still, he was able to find and dig out the treasure. How did he do that?)