# Fixed points

We'll prove that:

- To verify that two given isometries $\alpha$ and $\beta$ are the same, it is sufficient to verify that they agree on some three points that are not collinear.
- Every isometry is a product of at most three reflections.

Point $P \in \mathbb{R}^2$ is a **fixed point** of isometry $\alpha$ if $\alpha(P) = P$. We also say that $\alpha$ **fixes** point $P$.

**Lemma.** Let $\alpha$ be an isometry and $P$ a point such that $\alpha(P) \neq P$. Then every fixed point of $\alpha$ is on the perpendicular bisector between $P$ and $\alpha(P)$.

**Proof.**

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**Proof.**

**Theorem.** Let $\alpha$ be an isometry.

1. If $\alpha$ fixes three non-collinear points, then $\alpha = \iota$.
2. If $\alpha$ fixes two points then $\alpha = \iota$ or $\alpha$ is a reflection.
3. If $\alpha$ fixes some point then $\alpha$ is a product of at most two reflections.
4. Every isometry is a product of at most three reflections.

**Proof.**

**Corollary.** If $\alpha$ and $\beta$ are two isometries such that $\alpha(P) = \beta(P)$, $\alpha(Q) = \beta(Q)$ and $\alpha(R) = \beta(R)$, and points $P, Q$ and $R$ are not collinear, then $\alpha = \beta$.

**Proof.**

The proofs provide a simple method of finding the reflections when we know the images

$$\begin{aligned} P_0 &= \alpha(P), \\ Q_0 &= \alpha(Q), \text{ and} \\ R_0 &= \alpha(R) \end{aligned}$$

of three non-collinear points $P, Q$ and $R$.

We simply find reflections that **match the points one-by-one.**

**Corollary.** (This is Lemma 2.4 in the notes). If an isometry fixes two distinct points $P$ and $Q$, then it fixes every point of the line $m$ that contains $P$ and $Q$.

**Proof.**

## Symmetries of a set of points

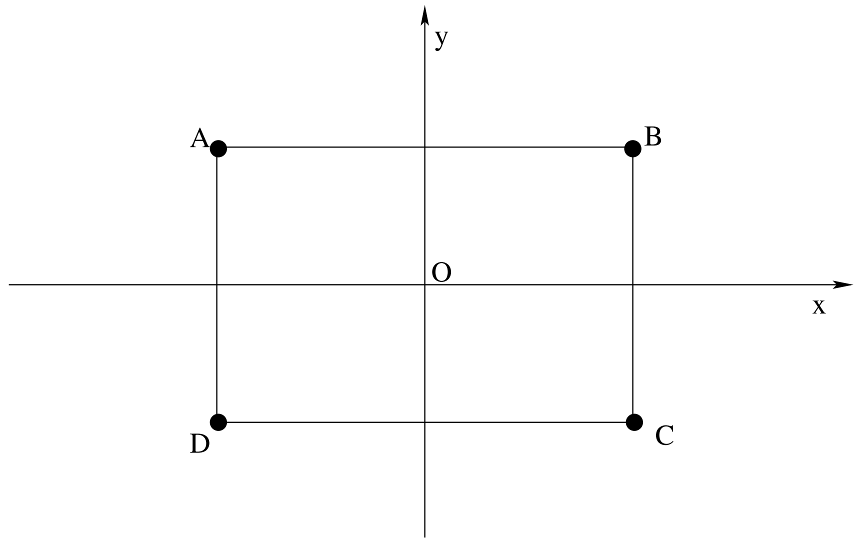
Isometry $\alpha$ is **symmetry** of a set $s \subseteq \mathbb{R}^2$ if $\alpha(s) = s$.

**Theorem.** For any $s \subseteq \mathbb{R}^2$ the symmetries of $s$ form a subgroup of $\mathcal{I}$.

The set of symmetries of $s$ is the **symmetry group** of $s$. Notice that $\mathcal{I}$ itself is the symmetry group of $s = \mathbb{R}^2$.

**Proof.**

**Example.** Let $s$ be a rectangle $ABCD$ that is not a square. Let us position $s$ in such a way that its center is at the origin $(0,0)$, and its sides are parallel to the $x$- and $y$-axes:

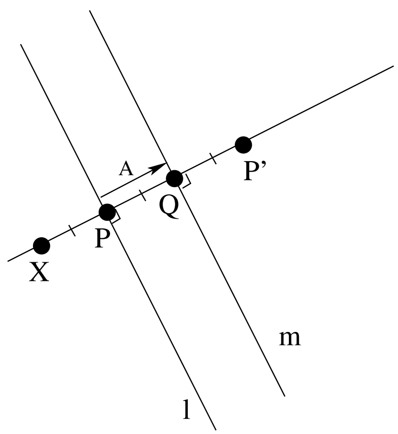



Let us find symmetries of $s$.

# Products of two reflections

**Theorem.** Let $m, \ell$ be parallel lines. Then $\sigma_m \sigma_\ell$ is the translation $\tau_{2A}$ where $A$ is the vector from $\ell$ to $m$ that is perpendicular to $\ell$ and $m$.

Conversely, every translation is a product of two reflections in parallel lines, both perpendicular to the direction of the translation. One of the lines can be chosen freely (as long as it is perpendicular to to the translation), after which the other line is uniquely determined.




**Proof.**

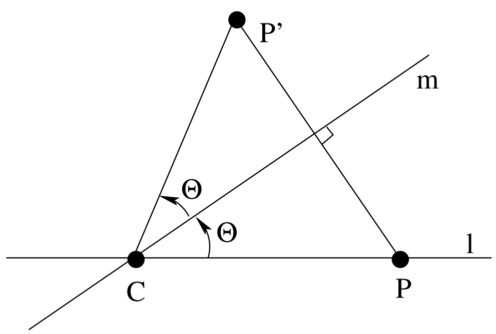
**Corollary.** The product of three reflections in three parallel lines is a reflection in a single parallel line.

**Proof.**

**Theorem.** Let $m, \ell$ be non-parallel lines, intersecting each other at point $C$. Then $\sigma_m \sigma_\ell$ is the rotation $\rho_{C,2\theta}$ where $\theta$ is the angle from line $\ell$ to line $m$.

Conversely, every rotation around $C$ is a product of two reflections in lines through point $C$. One of these lines can be chosen freely, after which the other line is uniquely determined.

**Proof.**



**Corollary.** Halfturn $\sigma_C$ is the product of two reflections in any two perpendicular lines through $C$. In particular, reflections in perpendicular lines commute.

**Corollary.** The product of three reflections in lines through the common point $C$ is a reflection in a line through point $C$.

**Proof.**