Products of three reflections

Lemma. The following three are equivalent for an isometry α:

1. α is a glide reflection,
2. $\alpha = \sigma_P \sigma_l$ for some point P and line l,
3. $\alpha = \sigma_k \sigma_Q$ for some line k and point Q.

Proof.

Theorem. A product of three reflections is a glide reflection.

Proof.

In the proof we use the following lemma:

Lemma (*). If m and l are two lines and P is a point, then there are lines p and q such that $\sigma_m \sigma_l = \sigma_p \sigma_q$, and line q contains point P.

Proof of the Lemma.

Theorem. A product of three reflections is a glide reflection.

Proof.

Corollary. Every plane isometry is a translation, a rotation or a glide reflection.

Proof. We know that every isometry is a product of at most three reflections.

- Products of three reflections are glide reflections.
- Products of two reflections are translations and rotations.
- Single reflections are also glide reflections.
- "The product of zero reflections" is the trivial isometry which is also a translation and a rotation.

Parity

All isometries are products of reflections. An isometry is

- **even** if it is a product of an even number of reflections,
- **odd** if it is a product of an odd number of reflections.

Let us show that every isometry is even or odd but not both: products of even and odd numbers of reflections can not be identical.

Theorem. A product of four reflections is a product of two reflections.

Proof.

Recall the following lemma.

Lemma (*) If m and l are two lines and P is a point, then there are lines p and q such that $\sigma_m \sigma_l = \sigma_p \sigma_q$, and line q contains point P.

In the proof of the theorem we use **Lemma (*)** twice.

Corollary. A product of an even number of reflections cannot equal a product of an odd number of reflections. Thus an isometry cannot be both even and odd.

Proof.

- Rotations and translations are the even isometries.
- Glide reflections are the odd isometries.

- Rotations and translations are the even isometries.
- Glide reflections are the odd isometries.

Even isometries form a subgroup of $\mathcal{I}$: composing n reflections with m reflections is a composition of $n+m$ reflections. And a sum of two even numbers is an even number.

Let us denote the group of even isometries by $\mathcal{E}$.

The set of odd isometries is not a group because a sum of two odd numbers is not odd. The set of odd isometries is the set $\gamma\mathcal{E}$, for any single odd isometry γ. In other words, it is a **coset** of the subgroup $\mathcal{E}$.

Products of even isometries

Theorem.

1. The product of two translations is a translation.
2. A rotation by angle Θ followed by a rotation by angle Φ is a rotation by angle $\Theta + \Phi$, unless $\Theta + \Phi$ is a multiple of $360°$, in which case the product is a translation.
3. A translation followed by a non-trivial rotation by Θ is a rotation by Θ. Also, a non-trivial rotation by Θ followed by a translation is a rotation by Θ.

In summary: Consider a translation as a "rotation by 0 degrees". Then composing rotations by angles $\theta_1, \theta_2, \ldots, \theta_n$ is a rotation by $\theta_1 + \theta_2 + \cdots + \theta_n$, unless this sum is a multiple of $360°$ in which case the result is a translation.

Proof.

Corollary. If a subgroup of $\mathcal{I}$ contains two non-trivial rotations about different centers then it also contains a non-trivial translation.

Proof.