Rosette groups

A finite subgroup of $\mathcal{I}$ is called a **Rosette group**.

Two families of Rosette groups: For every $n \geq 1$

- the **cyclic group**
$$C_n = \langle \rho \rangle = \{\rho^1, \rho^2, \ldots \rho^n = \iota\}$$
where ρ is a rotation by angle $\theta = 360°/n$, and
- the **dihedral group**
$$D_n = \{\rho^1, \rho^2, \ldots \rho^n = \iota\} \cup \{\rho^1\sigma, \rho^2\sigma, \ldots \rho^n\sigma = \sigma\}$$
where ρ is as above and σ is a reflection on a line through the center of rotation ρ.

Isometries $\rho^k\sigma$ in D_n are reflections in lines that meet at the center of ρ at angles that are multiples of
$$\frac{1}{2}\theta = 180°/n.$$
Both C_n and D_n are indeed groups.

Remark: Strictly speaking there are infinitely many groups C_n as the center P of ρ can be any point of the plane. Same for D_n: the center of ρ and the line of σ can vary.

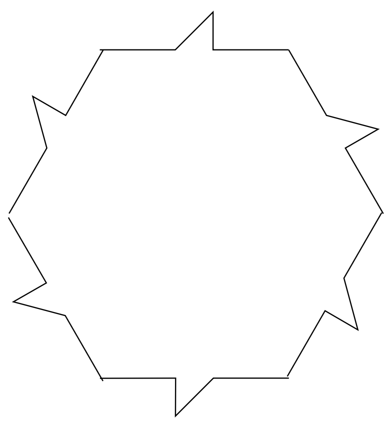
Example. Small $n = 1$ and 2:

- $C_1 = \{\iota\}$ and $D_1 = \{\iota, \sigma_m\}$,
- $C_2 = \{\iota, \sigma_P\}$ and $D_2 = \{\iota, \sigma_P, \sigma_m, \sigma_l\}$, where m and l are perpendicular lines through point P.

The symmetry group of a **polygon** is finite: A vertex must be mapped to a vertex, and adjacent vertices to adjacent vertices. Thus an n-gon has at most $2n$ symmetries.

Example. Consider a regular n-gon. It has n rotational symmetries, and also any reflection on a line through any vertex and the center of a polygon is a symmetry. Thus the symmetry group is the dihedral group D_n.

Example. A polygon with symmetry group C_n, $n \geq 3$:



Leonardo da Vinci's theorem. A finite subgroup of $\mathcal{I}$ is either a cyclic group C_n or a dihedral group D_n.

Proof.

Conjugacy

For any group G, elements $x, y \in G$ are **conjugate** if

$$\exists \alpha \in G \ : \ x = \alpha y \alpha^{-1}.$$

Conjugacy is an **equivalence relation.** Its equivalence classes are called the **conjugacy classes** of the group.

In the group $\mathcal{I}$ conjugate isometries are of the same type (both translations, both rotations, both reflections or both glide reflections).

Intuitively, $\alpha\beta\alpha^{-1}$ is map β done on the plane that has been transformed according to α.

Theorem. Let $\alpha \in \mathcal{I}$ be any isometry.

1. Let $\sigma = \sigma_m$ be the **reflection** in line m. Then $\alpha\sigma\alpha^{-1}$ is the reflection $\sigma_{\alpha(m)}$ in line $\alpha(m)$.
2. Let $\tau = \tau_{B-A}$ be the **translation** that moves point A to point $B = \tau(A)$. Then $\alpha\tau\alpha^{-1}$ is the translation $\tau_{\alpha(B)-\alpha(A)}$ that moves point $\alpha(A)$ to point $\alpha(B)$.
3. Let $\rho = \rho_{P,\Theta}$ be a **rotation** about point P. Then $\alpha\rho\alpha^{-1}$ is the rotation $\rho_{\alpha(P),\pm\Theta}$ about point $\alpha(P)$, where the angle is $+\Theta$ if α is even, and $-\Theta$ if α is odd.
4. Let $\gamma = \gamma_{m,B-A}$ be a **glide reflection.** Then $\alpha\gamma\alpha^{-1}$ is the glide reflection $\gamma_{\alpha(m),\alpha(B)-\alpha(A)}$.

Proof

1. Let $\sigma = \sigma_m$ be the **reflection** in line m. Then $\alpha\sigma\alpha^{-1}$ is the reflection $\sigma_{\alpha(m)}$ in line $\alpha(m)$.

2. Let $\tau = \tau_{B-A}$ be the **translation** that moves point A to point $B = \tau(A)$. Then $\alpha\tau\alpha^{-1}$ is the translation $\tau_{\alpha(B)-\alpha(A)}$ that moves point $\alpha(A)$ to point $\alpha(B)$.

3. Let $\rho = \rho_{P,\Theta}$ be a **rotation** about point P. Then $\alpha\rho\alpha^{-1}$ is the rotation $\rho_{\alpha(P),\pm\Theta}$ about point $\alpha(P)$, where the angle is $+\Theta$ if α is even, and $-\Theta$ if α is odd.

4. Let $\gamma = \gamma_{m,B-A}$ be a **glide reflection.** Then $\alpha\gamma\alpha^{-1}$ is the glide reflection $\gamma_{\alpha(m),\alpha(B)-\alpha(A)}$.

The theorem is useful in figuring out symmetry groups: If α, β are in a symmetry group so is the conjugate $\alpha\beta\alpha^{-1}$.

We use the following terminology. Let G be the symemtry group os $s \subseteq \mathbb{R}^2$.

- If $\sigma_m \in G$ then m is a **line of symmetry** of s.
- If $\sigma_P \in G$ then P is a **point of symmetry** of s.
- If $\rho_{C,\theta} \in G$ then C is a **center of symmetry** and, more precisely, if $\theta = \frac{360^\circ}{n}$ then C is a **center of n-fold symmetry** of s.

Let α be an arbitrary symmetry of set s. Then in s, using the theorem,

- If m is a line of symmetry then also $\alpha(m)$ is a line of symmetry.
- If P is a point of symmetry then also $\alpha(P)$ is a point of symmetry.
- If C is a center of (n-fold) symmetry then also $\alpha(C)$ is a center of (n-fold) symmetry.

Frieze groups

The set $\mathcal{T}$ of translations is a subgroup of $\mathcal{I}$. Thus for any group G of isometries the **translations in G** form a subgroup

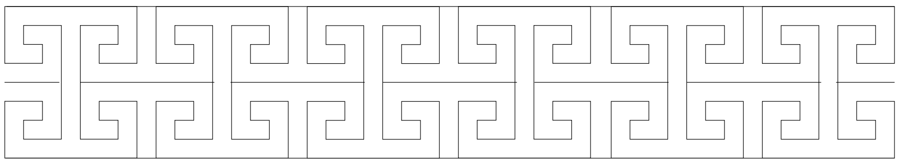
$$G \cap \mathcal{T}.$$

Group G is a **frieze group** if

$$G \cap \mathcal{T} = \langle \tau \rangle$$

for some non-trivial translation τ. So for some non-zero vector A the translations in G are precisely by multiples of A.

Example. The symmetry group of the bi-infinite horizontally repeating pattern



is a frieze group.

Example. The symmetry group of a horizontal line is not a frieze group although all translations are in the horizontal direction. However, there is no smallest positive translation.

There are **seven** different frieze groups (when we ignore the position, orientation and the size of the frieze).

Convention: The direction of the translations in the frieze group is fixed to be **horizontal**.