Wallpaper groups

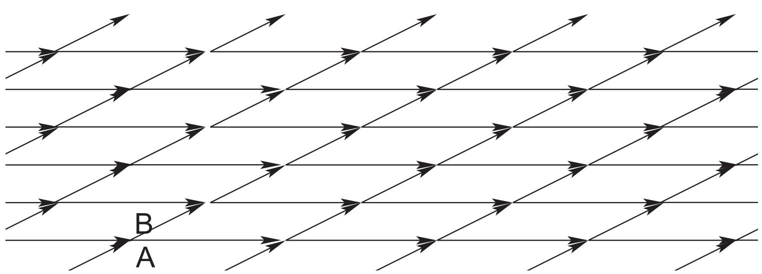
A subgroup of $\mathcal{I}$ is a **wallpaper group** if its translations are

$$G \cap \mathcal{T} = \langle \tau_1, \tau_2 \rangle,$$

where τ_1 and τ_2 are non-parallel translations.

Translations commute with each other, so the translations of G are exactly $\tau_1^i \tau_2^j$ for $i, j \in \mathbb{Z}$.

If A and B are the vectors of translations τ_1 and τ_2 then the vectors of translations $\tau_1^i \tau_2^j$ are $iA + jB$, which form a lattice



Lemma. A wallpaper group G has a shortest non-trivial translation. More generally, any non-empty subset s of translations of G contains a shortest non-trivial translation.

Proof.

An important restriction on possible rotations in wallpaper groups:

Theorem [Crystallographic restriction]. A wallpaper group G can only contain rotations by multiples of 60° and 90°. Hence all centers of rotations are centers of n-fold rotations for $n = 2, 3, 4$ or 6.

Moreover, a 4-fold rotation cannot co-exist with 3- or 6-fold rotations.

Proof.

A subgroup G of $\mathcal{I}$ is **discrete** if $\exists \varepsilon > 0$ such that

$$\begin{aligned} 0 < |A| < \varepsilon &\implies \tau_A \notin G, \text{ and} \\ 0 < \Theta < \varepsilon &\implies \rho_{C,\Theta} \notin G. \end{aligned}$$

($|A|$ is the length of the translation vector A.)

In other words: a discrete group G does not contain arbitrarily short translations and does not contain arbitrarily small rotations.

Lemma. Let G be a discrete subgroup of $\mathcal{I}$ that contains translations in non-parallel directions. Let

- τ_A be a shortest non-zero translation in G, and
- let $\tau_B \in G$ be a shortest translation not generated by τ_A.

Then τ_A and τ_B generate all translations of G.

Proof.

Theorem. Discrete subgroups of $\mathcal{I}$ are exactly the rosette groups, frieze groups and wallpaper groups.

Proof.