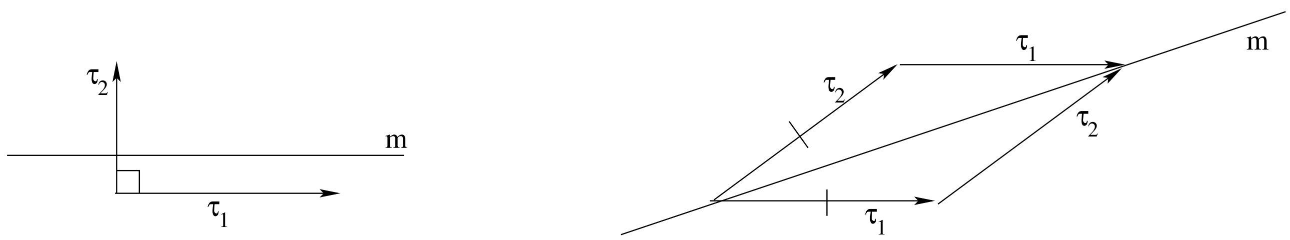
Lemma. Let G be a wallpaper group that contains an odd isometry with axis m. Then there exist translations $\tau_1, \tau_2 \in G$ that generate all translations of G and either

(1) τ_1 is parallel to m and τ_2 is perpendicular to m, or

(2) τ_1 and τ_2 are of equal length and m is parallel to $\tau_1\tau_2$.

Moreover, in case (2), group G contains a reflection.



Case (1) Case (2)

In case (1) the translation lattice is rectangular, and m is parallel to a side of the rectangles, and in case (2) the translation lattice is rhombic, and m is parallel to a diagonal of the rhombi.

Remark. Each rosette, frieze or wallpaper group type is actually a **family** of subgroups of $\mathcal{I}$. Depending on the center of rotations, the generating translations etc., we get different groups but they have the same type.

More precisely, each group type represents a family of **affinely conjugate** subgroups.

An **affine** transformation of the plane is a transformation that preserves parallelism of lines. It is the composition of a linear transformation and a translation:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

where M is a 2×2 matrix.

The transformation is **one-to-one** if and only if M is invertible, i.e., $\det(M) \neq 0$.

Isometries are exactly the distance preserving affine maps. Distance preservation is equivalent to M being an **orthogonal matrix**, i.e., equivalent to

$$MM^T = I$$

where M^T is the transpose of M and I is the 2×2 identity matrix.

Even and odd isometries correspond to orthogonal matrices M whose determinant is $+1$ and -1, respectively.

Two subgroups G_1 and G_2 of $\mathcal{I}$ are said to be equal up to **affine conjugacy** if there exists a one-to-one affine transformation f such that

$$G_1 = fG_2f^{-1}.$$

In particular, this requires that $f\alpha f^{-1}$ are isometries for all $\alpha \in G_2$ (which is **not** the case for all affine f and all isometries $\alpha \in \mathcal{I}$).

If G_1 and G_2 are wallpaper groups, frieze groups or rosette groups then equality up to affine conjugacy exactly means that they are of the same wallpaper, frieze or rosette group type.

Affine conjugacy **preserves isometry types**: If α and $f\alpha f^{-1}$ are both isometries then they are of the same type: both translations, both rotations, both reflections or both glide reflections.

(To see this, note that the parity of the isometry is preserved by affine conjugacy, and that P is a fixed point of α if and only if $f(P)$ is a fixed point of $f\alpha f^{-1}$.)

As groups, C_2 **and** D_1 are isomorphic. But they are not equal up to affine conjugacy.

Likewise, frieze groups F_{0000} **and** F_{0001} are isomorphic (both are infinite cyclic groups, one is generated by a translation the other one by a glide reflection) but we consider them different as they are not affinely conjugate.

Recall basic topological concepts of $\mathbb{R}^2$

- **open** and **closed** sets,
- **neighborhood** of a point (=any open set containing the point),
- **interior** of a set (=largest open set contained in the set),
- **closure** of a set (=smallest closed set containing the set),
- **boundary** of a set (=intersection of the closures of the set and its complement),
- **compactness** (in $\mathbb{R}^2$ this means closed and bounded),

Recall basic topological concepts of $\mathbb{R}^2$

- **continuity** of functions (inverse images of open sets are open),
- **homeomorphism** (=continuous bijection whose inverse is also continuous).
- **connectedness** (an open set is connected iff it is not the union of two disjoint open sets),

In $\mathbb{R}^2$ an open set is connected if and only if it is path-connected: each pair of its points can be joined by a path (=homeomorphic image of the unit interval) inside the set.

We denote

$$B_r(P) = \{X \in \mathbb{R}^2 \mid d(X, P) < r\}$$

for the **open disk** (or ball) of radius r centered at P.

We also denote $B_r = B_r(0, 0)$.

The closure of an open disk is a **closed disk:**

$$\overline{B}_r(P) = \{X \in \mathbb{R}^2 \mid d(X, P) \leq r\},$$

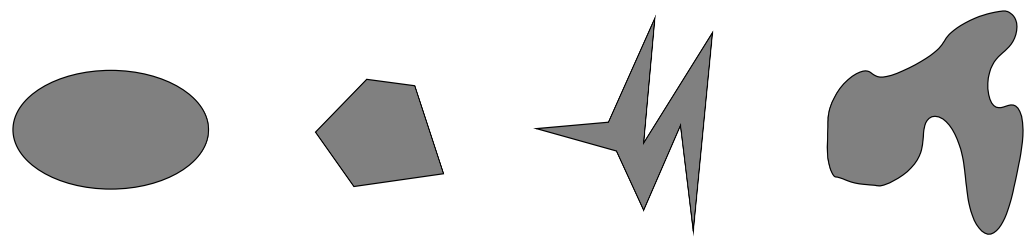
and $\overline{B}_r = \overline{B}_r(O)$.

A **tile** is a subset of $\mathbb{R}^2$ that is a **topological disk**, that is, it is the image of the closed disk $\overline{B}_1$ under some homeomorphism.

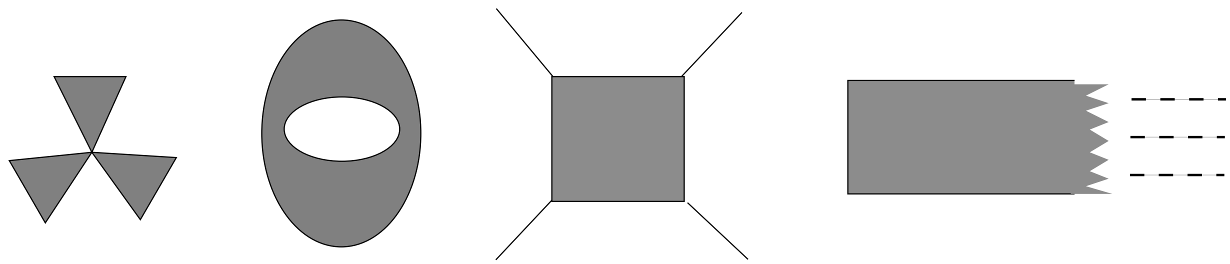
Homeomorphisms preserve topological properties, so tile t inherits topological properties from the disk $\overline{B}_1$:

- t is compact (=closed and bounded),
- the interior of t is connected, and the complement of t is connected,
- the boundary of t is the boundary of its interior,
- the boundary of t is a simple closed curve, that is, homeomorphic to the unit circle
$$\{X \in \mathbb{R}^2 \mid d(X, O) = 1\}.$$

Here are some examples of tiles:



These are **not** tiles:



A **tiling** $\mathcal{T}$ is a family of tiles that covers the plane

(1) **without gaps** (every $P \in \mathbb{R}^2$ belongs at least one tile), and

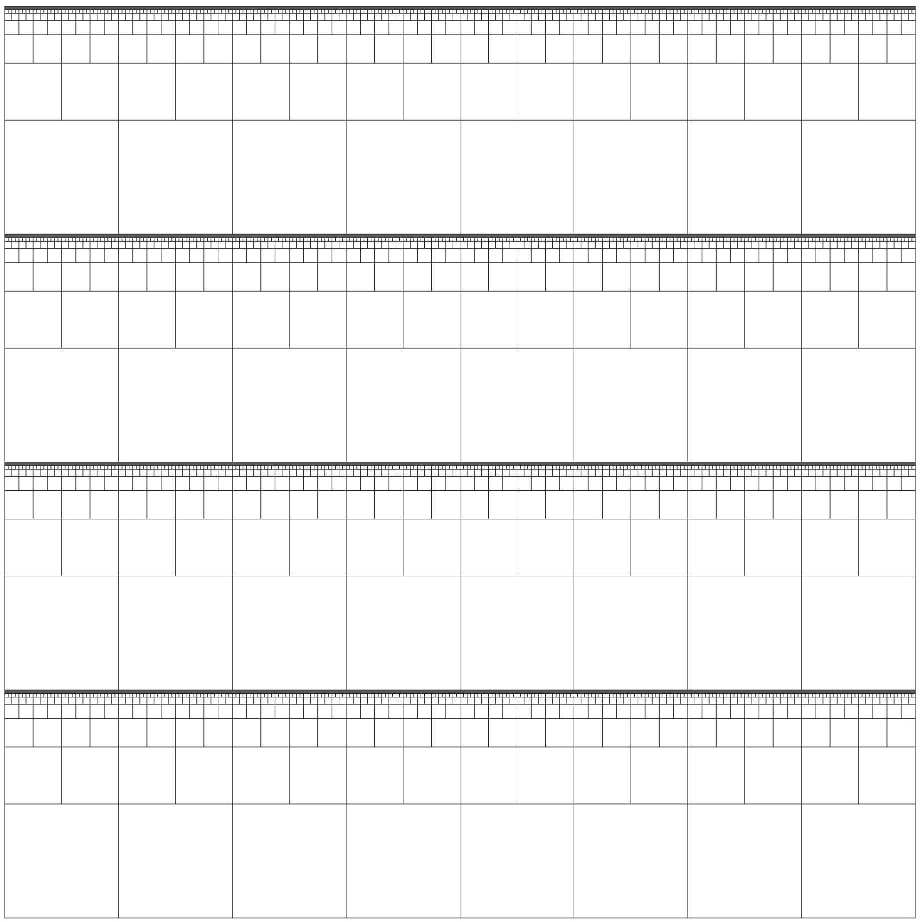
(2) **without overlaps** (the interiors of the tiles are pairwise disjoint).

The boundaries of the tiles do not need to be disjoint. But any interior point of a tile cannot belong to any other tile.

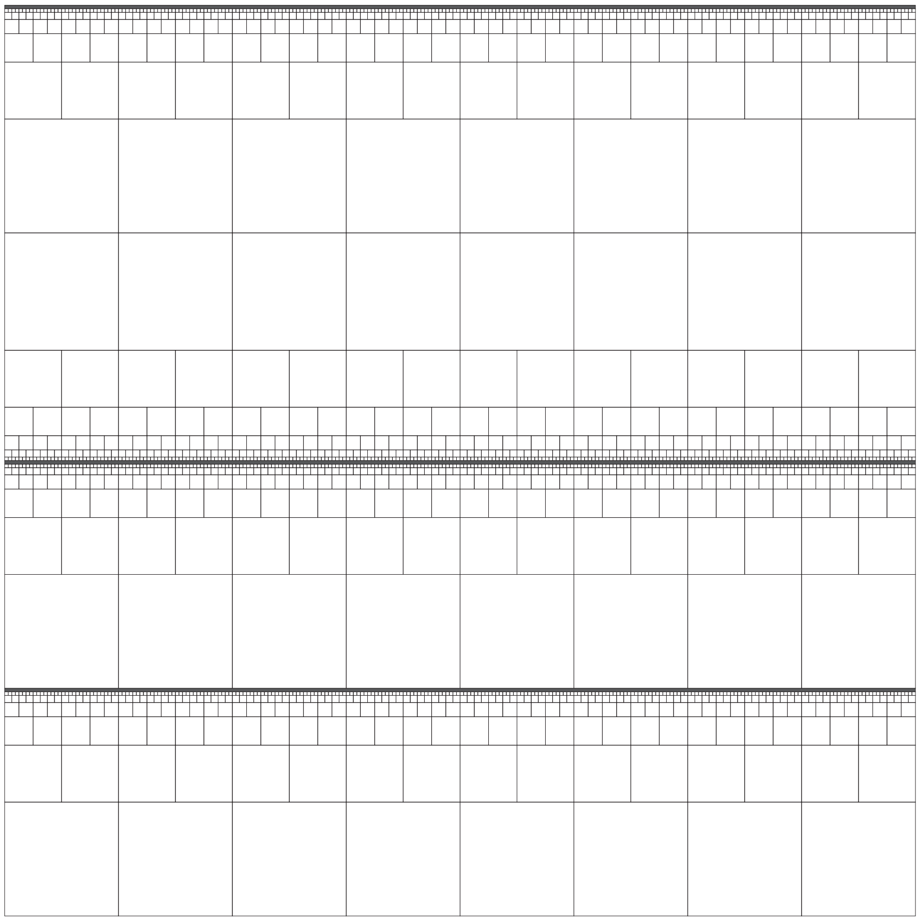
Remark: The number of tiles in any tiling must be

- infinite (union of a finite number of bounded sets would be bounded), but
- countable (the interior of each tile contains a point with rational coordinates).

This **is** (part of) a tiling:



But this (with one strip flipped) **is not**:



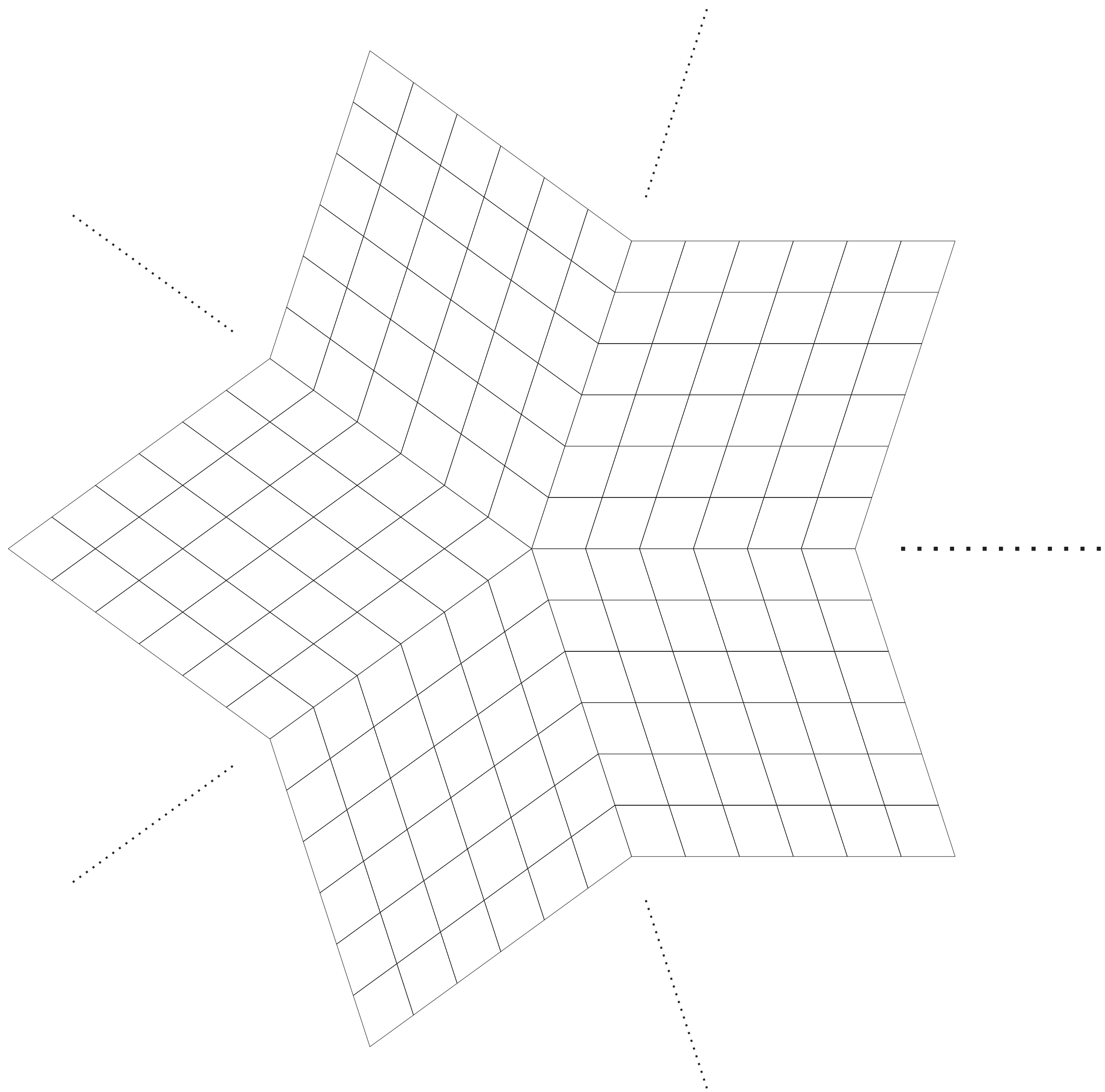
Let $\mathcal{T} = \{t_1, t_2, \dots\}$ be a tiling. Its **symmetry group** G consists of those isometries α that take every tile of $\mathcal{T}$ onto a tile of $\mathcal{T}$:

$$\forall i \ : \ \exists j \ : \ \alpha(t_i) = t_j.$$

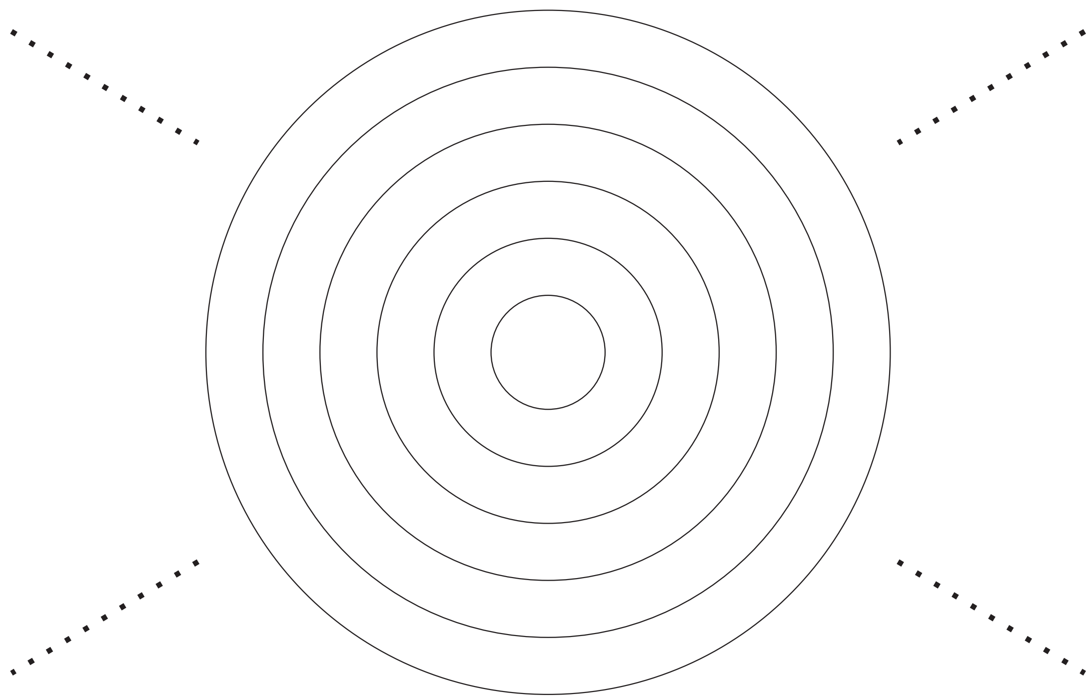
Theorem. The symmetry group of a tiling is discrete.

Proof.

Example. This tiling has symmetry group D_5:



Remark: This "tiling" has a non-discrete symmetry group. But it is **not** a tiling!



Usually we want only finite number of different shapes in our tilings.

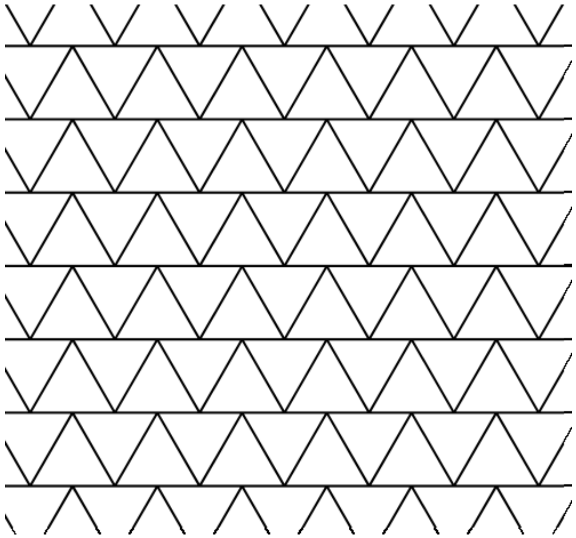
We call two tiles t, t' **congruent** if $t' = \alpha(t)$ for some isometry α.

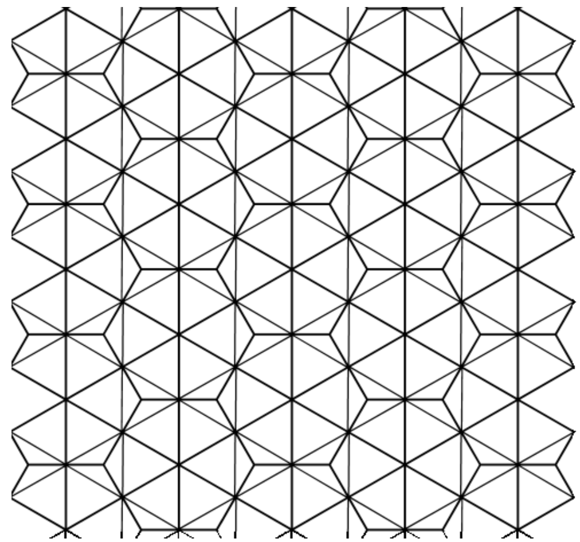
Tiles $\{p_1, p_2, \ldots, p_k\}$ are **prototiles** of a tiling $\mathcal{T} = \{t_1, t_2, \ldots\}$ if every tile $t_i \in \mathcal{T}$ is congruent to some p_j.

Tiling $\mathcal{T}$ is ***k*-hedral** where k is the number of prototiles p_j. (And in in the special cases of $k = 1$ and $k = 2$ the tiling is called **monohedral** and **dihedral**.)

Remark: Tiles may be "flipped over" copies of the prototiles since odd isometries are allowed in the definition of congruence of tiles.

Example. a monohedral and a dihedral tiling:





Two tiles t_1 and t_2 of tiling $\mathcal{T}$ are **transitive** (or **equivalent**) in $\mathcal{T}$ if there exists a symmetry of $\mathcal{T}$ that takes t_1 onto t_2. This is an **equivalence relation** among tiles t_i. Equivalence classes are called the **transitivity classes** of $\mathcal{T}$.

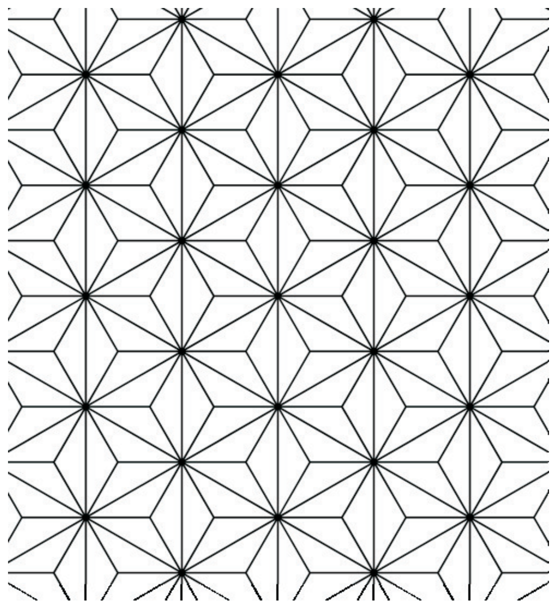
If tiling $\mathcal{T}$ has only one transitivity class then the tiling is called **isohedral** (or **tile-transitive**).

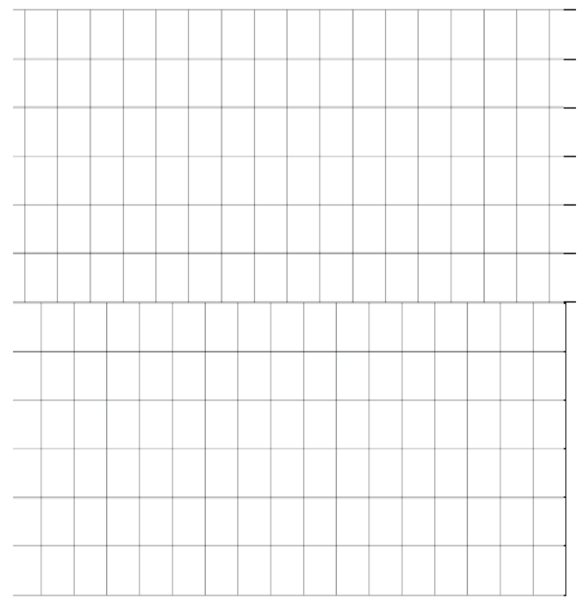
If there are k transitivity classes then the tiling is called ***k*-isohedral**.

Of course, any isohedral tiling is monohedral as transitive tiles are congruent. But there are monohedral tilings that are not isohedral.

Analogously, a k-isohedral tiling is always k-hedral (but it can also be n-hedral for some $n < k$).

Example. An isohedral tiling and a monohedral tiling that is not isohedral (not even k-isohedral for any finite k).





The symmetry group of a k-hedral tiling is a wallpaper group if and only if the tiling is n-isohedral for some n (homework).

Let

$$\mathcal{T} = \{t_1, t_2, t_3, \ldots\}$$

be a tiling. If $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a homeomorphism then also

$$h(\mathcal{T}) = \{h(t_1), h(t_2), h(t_3), \ldots\}$$

is a tiling. We say that tilings $\mathcal{T}$ and $h(\mathcal{T})$ are **topologically equivalent**. This is an equivalence relation among tilings.

Every isometry is a homeomorphism, so if α is an isometry then $\alpha(\mathcal{T}) = \{\alpha(t_1), \alpha(t_2), \alpha(t_3), \ldots\}$ is a tiling. We say that that $\alpha(\mathcal{T})$ is **congruent** to tiling $\mathcal{T}$. Also congruence is an equivalence relation among tilings.

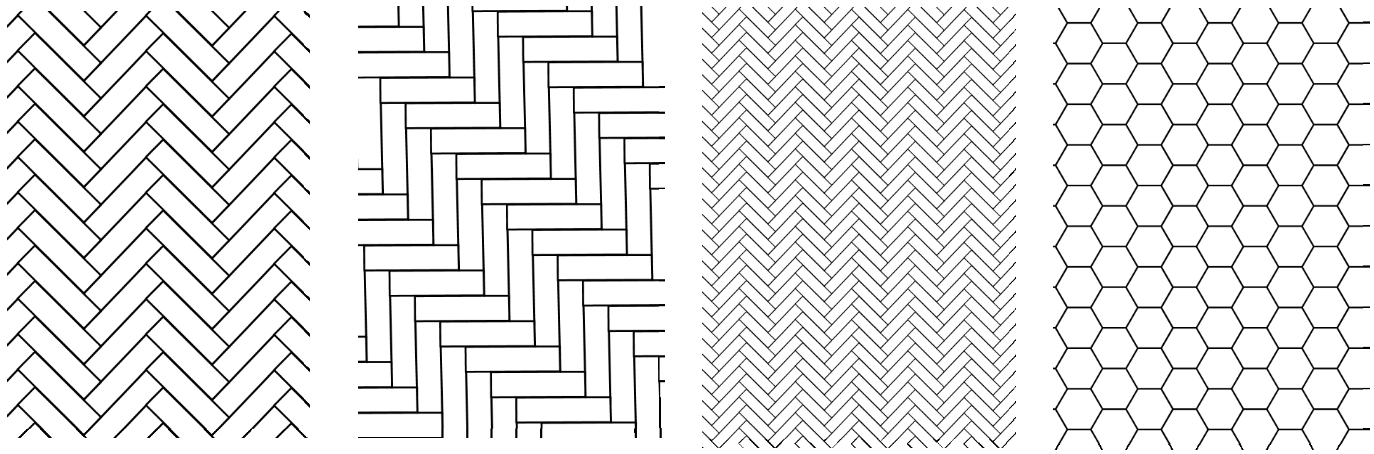
A **similarity** $s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a composition of an isometry and a **stretch** map

$$(x, y) \mapsto (kx, ky)$$

for some $k > 0$. We say that tilings $\mathcal{T}$ and $s(\mathcal{T})$ are **similar**.

Similarity of two tiling means that they look the same when one of them is watched under a suitable magnifying class. Usually we consider similar tilings to be the same tiling.

Example. Four topologically equivalent monohedral tilings. First two are congruent with each other, and they are similar to the third one:



Tilings by regular polygons

A tiling whose tiles are polygons is **edge-to-edge** if the intersection of two tiles is either empty, a vertex of the polygons, or the entire edges of the two neighboring polygons.

Two tiles are

- **edge neighbors** if their intersection is an edge of both polygons,
- **vertex neighbors** if their intersection is a vertex of both polygons.

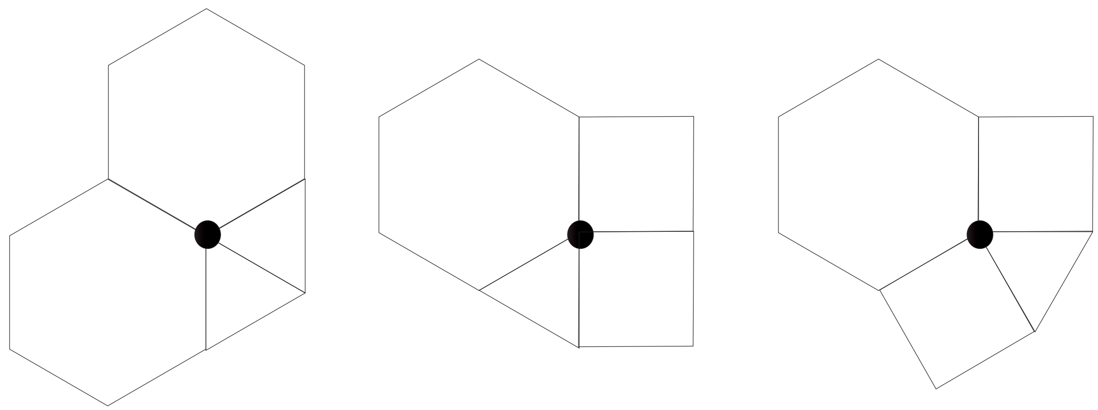
Consider a vertex P where r **regular polygons** of orders

$$n_1, n_2, \ldots, n_r$$

meet, in this order (counted clockwise or counterclockwise). We say that the vertex is of **type**

$$n_1 \cdot n_2 \cdot \cdots \cdot n_r.$$

Example. Vertices of types $3 \cdot 3 \cdot 6 \cdot 6$, $3 \cdot 4 \cdot 4 \cdot 6$ and $3 \cdot 4 \cdot 6 \cdot 4$:



Remark: Types $3 \cdot 4 \cdot 4 \cdot 6$ and $4 \cdot 6 \cdot 3 \cdot 4$ and $4 \cdot 3 \cdot 6 \cdot 4$ are all identical, as they are obtained by changing the starting point and/or the direction of reading the polygons.

We also adapt a shorthand notations for repetitions:
$3 \cdot 3 \cdot 6 \cdot 6$ may be abbreviated as $3^2 \cdot 6^2$.

The interior angle of a regular n-gon is

Consequently, if P is a vertex of type $n_1 \cdot n_2 \cdot \cdots \cdot n_r$ then

$$\sum_{i=1}^{r} \left(1 - \frac{2}{n_i}\right) =$$

(The interior angles of the polygons that meet at P must sum up to 360°.)

This limits the possible vertex types: only finitely many possibilities remain.

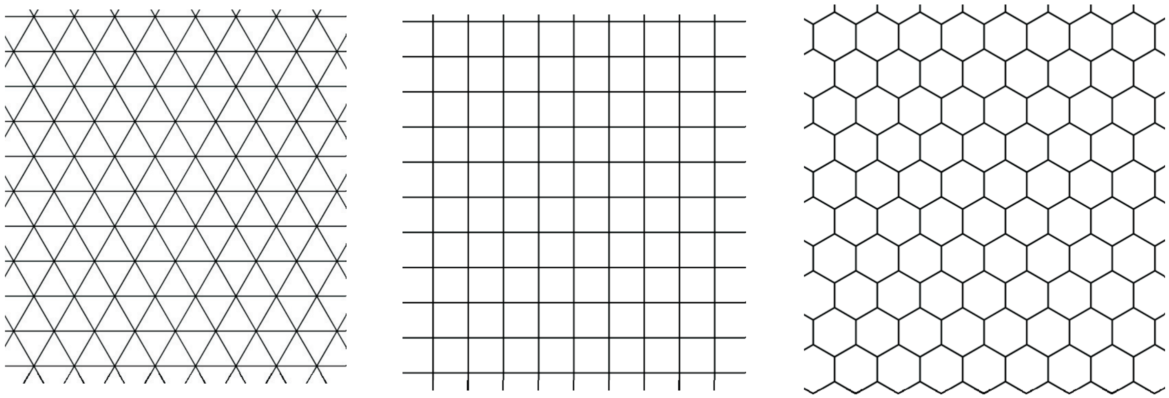
$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right)=2$$

Suppose first a vertex type n^r, where r copies of regular n-gons meet. We then get the condition

$$n = \frac{2r}{r-2}$$

Because n is positive, we have $r \geq 3$, and because $n \geq 3$ we have $r \leq 6$.

With $r = 3, 4, 5$ and 6 we get $n = 6, 4, \frac{10}{3}$ and 3. Number n is an integer so we only have three solutions. These vertex types appear in the familiar **regular** tilings



Theorem. The only edge-to-edge monohedral tilings by regular polygons are the three regular tilings above.

$$\sum_{i=1}^{r} \left(1 - \frac{2}{n_i}\right) = 2$$

Theorem. There are only finitely many vertex types.

$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right)=2$$

Theorem. There are only finitely many vertex types.

Proof. It is enough to show that there are finitely many solutions that satisfy

$$n_1 \leq n_2 \leq \cdots \leq n_r.$$

We have $r \geq 3$ and $n_i \geq 3$, which implies that $r \leq 6$.

Also $n_{r-1} \leq 12$: If $n_{r-1} > 12$ then also $n_r > 12$ so

$$\sum_{i=1}^{r}\left(1-\frac{2}{n_i}\right) \geq \left(1-\tfrac{2}{n_1}\right)+\left(1-\tfrac{2}{n_{r-1}}\right)+\left(1-\tfrac{2}{n_r}\right)$$

$$> \left(1-\tfrac{2}{3}\right)+\left(1-\tfrac{2}{12}\right)+\left(1-\tfrac{2}{12}\right)$$

a contradiction.

There are only finitely many tuples $(n_1, n_2, \ldots, n_{r-1})$ with $r \leq 6$ and $3 \leq n_1 \leq \cdots \leq n_{r-1} \leq 12$. The last number n_r is uniquely determined by $n_1, n_2, \ldots, n_{r-1}$ so there are finitely many vertex types.

The possible vertex types:

type	archimedean
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	A
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$	A
$3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$	A
$3 \cdot 3 \cdot 4 \cdot 3 \cdot 4$	A
$3 \cdot 3 \cdot 4 \cdot 12$	
$3 \cdot 3 \cdot 6 \cdot 6$	
$3 \cdot 4 \cdot 3 \cdot 12$	
$3 \cdot 4 \cdot 4 \cdot 6$	
$3 \cdot 4 \cdot 6 \cdot 4$	A
$3 \cdot 6 \cdot 3 \cdot 6$	A
$3 \cdot 7 \cdot 42$	
$3 \cdot 8 \cdot 24$	
$3 \cdot 9 \cdot 18$	
$3 \cdot 10 \cdot 15$	
$3 \cdot 12 \cdot 12$	A
$4 \cdot 4 \cdot 4 \cdot 4$	A
$4 \cdot 5 \cdot 20$	
$4 \cdot 6 \cdot 12$	A
$4 \cdot 8 \cdot 8$	A
$5 \cdot 5 \cdot 10$	
$6 \cdot 6 \cdot 6$	A

An edge-to-edge tiling by regular polygons is **archimedean** if all vertices of the tiling are of the same type.

The three regular tilings are all archimedean, corresponding to vertex types 6^3, 4^4 and 3^6.

There are only eight non-regular archimedean tilings, corresponding to the vertex types marked by "A" in the table.