Tiling problem with a seed tile

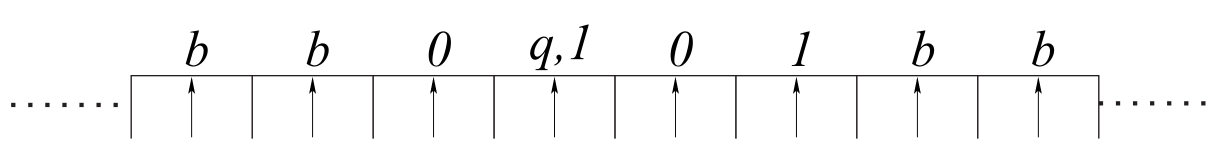
We associate to each Turing machine M a Wang tile set $\mathcal{P}_M$ whose tilings simulate iterations of M.

Horizontal rows of tilings represent configurations on consecutive time instances, time increasing upwards.

Colors will be represented as labeled arrows: In neighboring tiles arrow heads and tails with identical labels must meet each other.

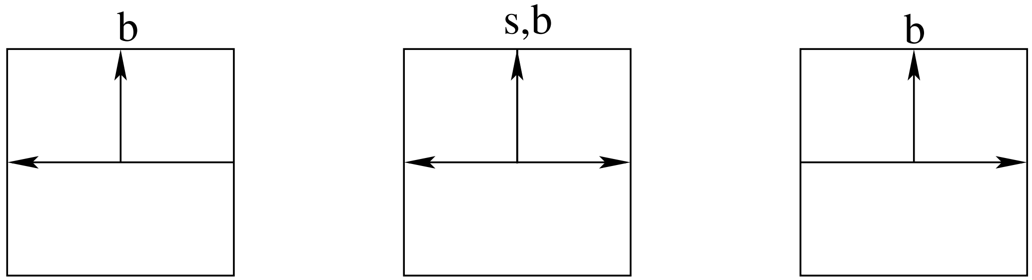
The labels are

- tape symbols (representing a tape location containing that symbol), or
- state/tape symbol pairs (representing a tape location containing the control unit at the given state).



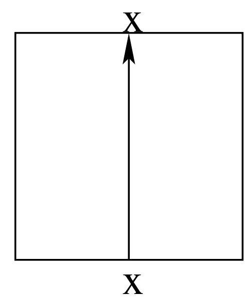
Let $M = (S, \Gamma, \delta, s, h, b)$. In $\mathcal{P}_M$ we have the following tiles:

(i) three starting tiles to represent the blank tape



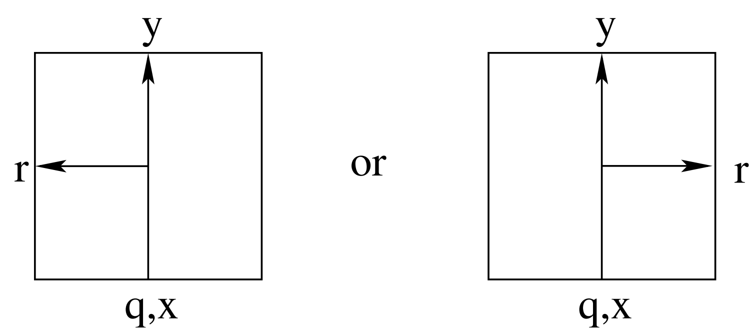
Let $M = (S, \Gamma, \delta, s, h, b)$. In $\mathcal{P}_M$ we have the following tiles:

(ii) for every tape letter $x \in \Gamma$ an alphabet tile



Let $M = (S, \Gamma, \delta, s, h, b)$. In $\mathcal{P}_M$ we have the following tiles:

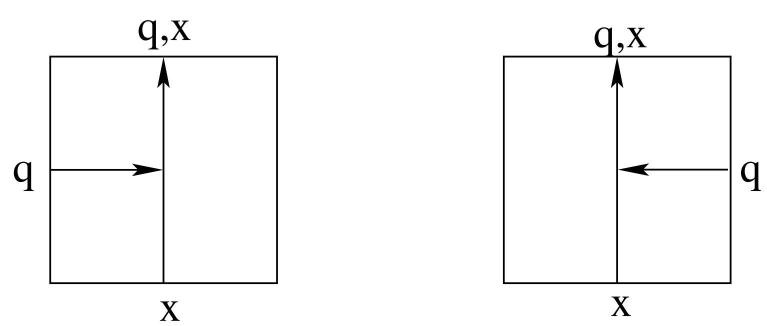
(iii) for every non-halting state $q \in S \setminus \{h\}$ and tape symbol $x \in \Gamma$ one action tile



where the left tile is used iff $\delta(q, x) = (r, y, L)$ and the right tile iff $\delta(q, x) = (r, y, R)$,

Let $M = (S, \Gamma, \delta, s, h, b)$. In $\mathcal{P}_M$ we have the following tiles:

(iv) for every non-halting state $q \in S \setminus \{h\}$ and tape symbol $x \in \Gamma$ the two merging tiles



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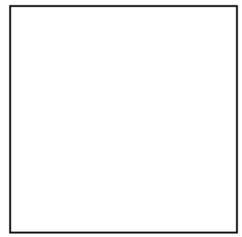
(v) the blank tile



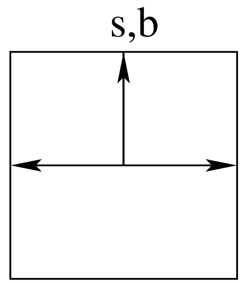
Seeded tiling problem

Instance: a finite set P of Wang prototiles and one specified seed tile $t \in P$

Positive instance: P and t such that there exists a valid tiling that contains t

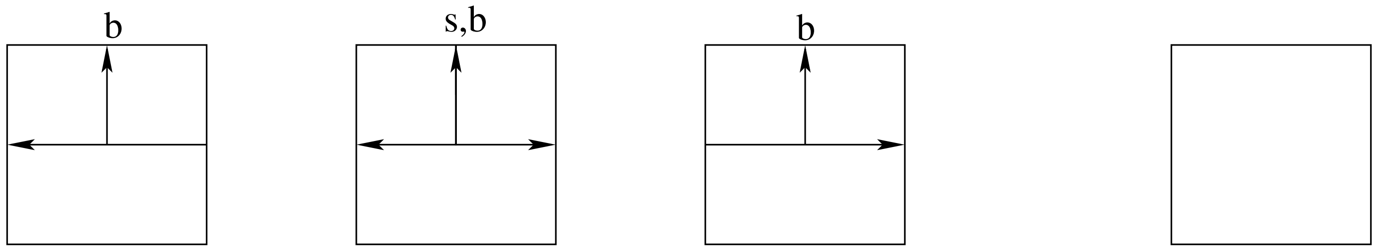
Theorem. Seeded tiling problem is undecidable.

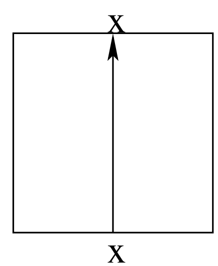
Proof. Reduction from **Halting from blank tape**: For a given TM M construct the tile set P_M and set

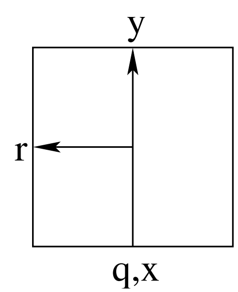


as the seed tile. Tile set P_M can be effectively constructed. It is now enough to note that P_M admits a tiling containing the seed tile if and only if M does not halt from the empty initial tape.

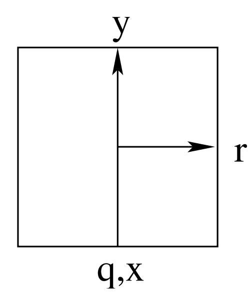
$$M = (S, \Gamma, \delta, s, h, b)$$

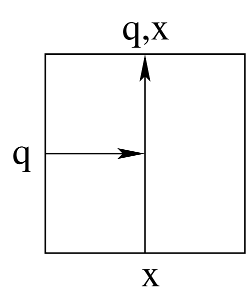


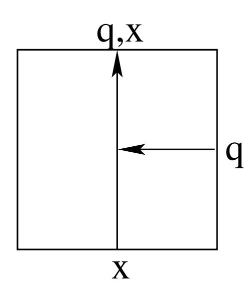
$\forall x \in \Gamma$: 

$\forall x \in \Gamma, \forall q \in S \setminus \{h\}$: 

or



$\forall x \in \Gamma, \forall q \in S \setminus \{h\}$: 



Note: **Seeded tiling problem** is not the same problem as **Tiling problem**.

The request for the named seed tile to appear in the tiling makes the proof easy, as it allows to guarantee the proper initialization of the Turing machine computation.

Undecidability of **Tiling problem** is significantly harder to prove.

Finite systems of forbidden patterns

(aka **subshifts of finite type**)

To make constructing tile sets easier we may replace the edge coloring based matching condition of Wang tiles by specifying a finite collection of **forbidden patterns**. A configuration is then a valid tiling if and only if it does not contain a forbidden pattern.

Take

- a finite set A of symbols,
- a finite $N \subseteq \mathbb{Z}^2$, a **neighborhood**,
- a set $R \subseteq A^N$ of **allowed patterns**

The complement $F = A^N \setminus R$ is the corresponding set of **forbidden patterns**.

A configuration $c \in A^{\mathbb{Z}^2}$ is in the **SFT (subshift of finite type)** defined by allowed patterns $R \subseteq A^N$ (or forbidden patterns $F \subseteq A^N$) if for every $\vec{a} \in \mathbb{Z}^2$

$$\tau_{\vec{a}}(c)|_N \in R,$$

where $\tau_{\vec{a}}$ is the translation of the configuration by vector $\vec{a}$:

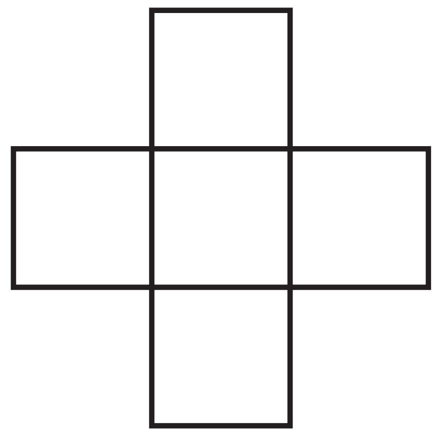
$$\forall \vec{b} \in \mathbb{Z}^2 \quad \tau_{\vec{a}}(c)(\vec{b}) = c(\vec{b} - \vec{a}).$$

In other words: c does not contain any of the forbidden patterns in any position $\vec{a}$.

Tilings by any Wang tile set can be expressed as an SFT with the neighborhood

$$N = \{(0,0), (0,-1), (0,1), (-1,0), (1,0)\},$$

and the set R that contains all those cross shaped patterns where the colors of the neighboring tiles match.



There is also a correspondence to the **other direction**: for any given SFT we can effectively (=algorithmically) construct a Wang tile set T such that there is a natural correspondence between configurations in the SFT and valid tilings by T.

Let $R \subseteq A^N$ be the given **allowed** patterns. We construct Wang tiles as follows:

(i) First note that if $N \subseteq N'$ for another neighborhood N' then we can construct allowed patterns $R' \subseteq A^{N'}$ so that R and R' define the same SFT.

Indeed: we take in R' all patterns $p \in A^{N'}$ such that $p|_N \in R$.

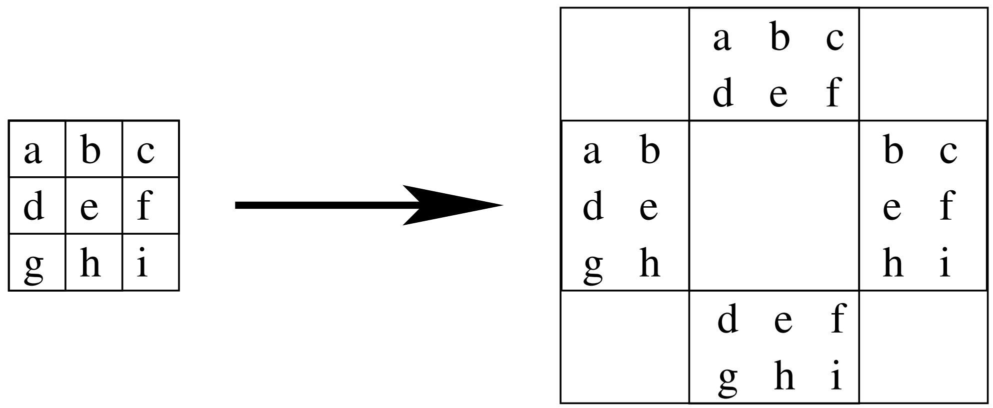
Because N is a subset of some square N', we thus convert R into an equivalent set R' of square shapes patterns.

Let $R \subseteq A^N$ be the given **allowed** patterns where N is an $m \times m$ **square**.

(ii) Construct a set P of Wang tiles as follows: The tiles are the allowed $m \times m$ square patterns over A, that is, $P = R$.

Colors of $t \in R$ are obtained by erasing one boundary column or row from it:

For example, the following figure illustrates the tile corresponding to a 3×3 allowed pattern:



Allowed 3x3 pattern   Wang tile

Two adjacent Wang tiles match if and only if the $m \times m$ patterns they represent have the correct $(m-1) \times m$ or $m \times (m-1)$ overlap when the tiles are placed next to each other.

Let $c \in A^{\mathbb{Z}^2}$ be any configuration. Associate to it the configuration

$$\Psi(c) \in (A^{m \times m})^{\mathbb{Z}^2}$$

where for each $(i, j) \in \mathbb{Z}^2$ we have in position (i, j) the $m \times m$ block that appears in c in position (i, j) (lower left corner at position (i, j).) This is known as a **higher block presentation** of c.

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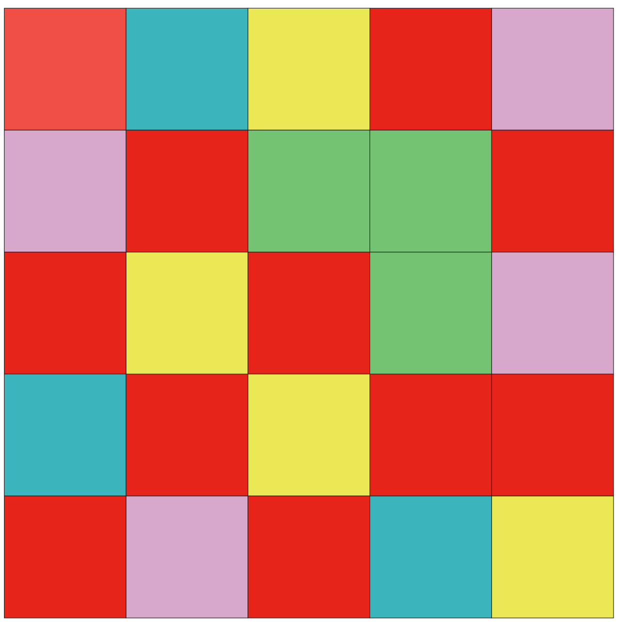
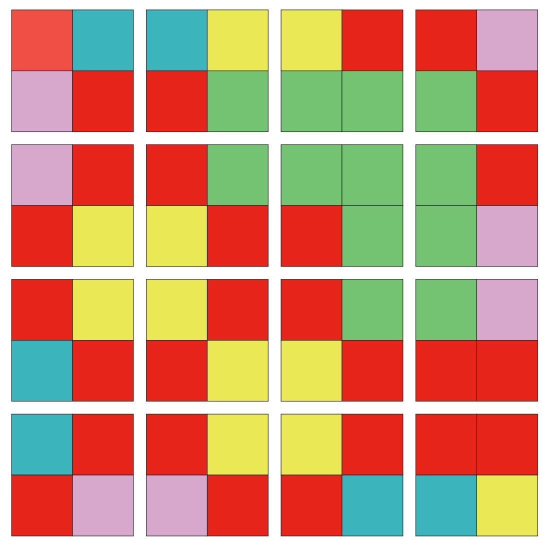


c



$\psi(c)$
m=2

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• The map $c \mapsto \Psi(c)$ is one-to-one:

• If c only contains allowed patterns then $\Psi(c) \in P^{\mathbb{Z}^2}$ and neighboring tiles match in color. So $\Psi(c)$ is a valid tiling by Wang prototiles P.

• Conversely, any valid tiling t is $\Psi(c)$ for a suitable c: Such c is obtained by pasting, for each $(i, j) \in \mathbb{Z}^2$, tile $t(i, j)$ as the $m \times m$ block in c with lower left corner in position (i, j). Clearly then c only contains allowed patterns since all pasted patterns are allowed patterns.

Note: c is well-defined because the pasted $m \times m$ blocks are compatible; if two pasted blocks overlap they assign the same symbols in the overlapping region.

In summary, we have the following directions:

(i) For every Wang protoset P one can effectively construct an SFT over P such that $c \in P^{\mathbb{Z}^2}$ is in the SFT if and only if c is a valid Wang tiling.

(ii) Conversely, for every SFT one can effectively construct a Wang protoset P such that P admits a (periodic) tiling if and only if the SFT contains a (periodic) tiling.

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In the following reductions we can describe tiles in any terms that locally determine which tiles are allowed to be next to each other. Such tiles can anyway be effectively converted into an equivalent set of Wang tiles. This substantially simplifies the constructions.