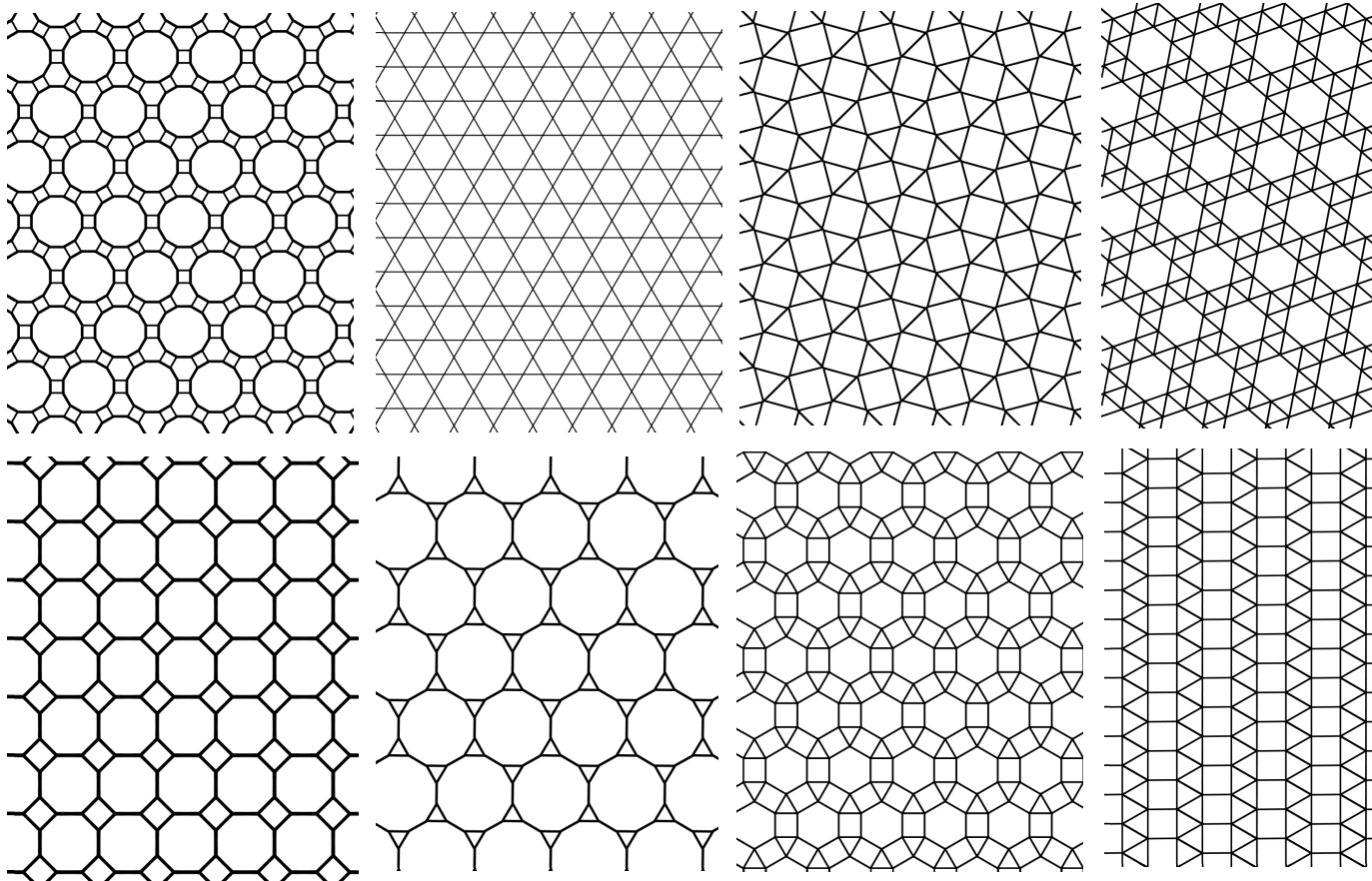


Theorem [Kepler 1619]. There are exactly eleven different archimedean tilings, one of each type indicated by "A" in the table.



Proof. We show that

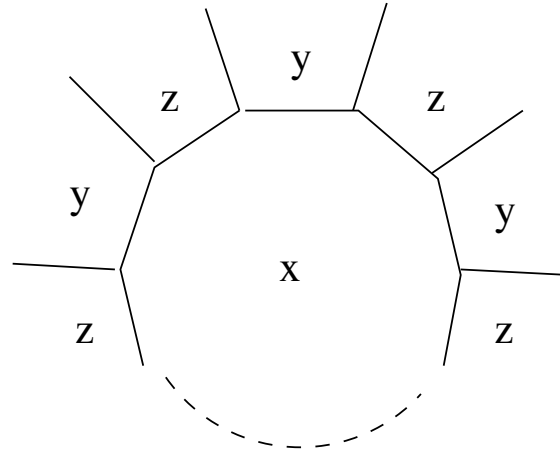
- (i) the vertex types without "A" in the table are not possible in archimedean tilings, and
- (ii) each type with "A" leads to a unique tiling (up to similarity).

Some terminology:

A polygon is **incident** to its vertices and edges, and an edge is **incident** to its endpoints.

Two vertices are **neighbors** if they are the two endpoints of an edge.

(i) Vertex type $x \cdot y \cdot z$ where x is odd and $y \neq z$ is not possible in any archimedean tiling.



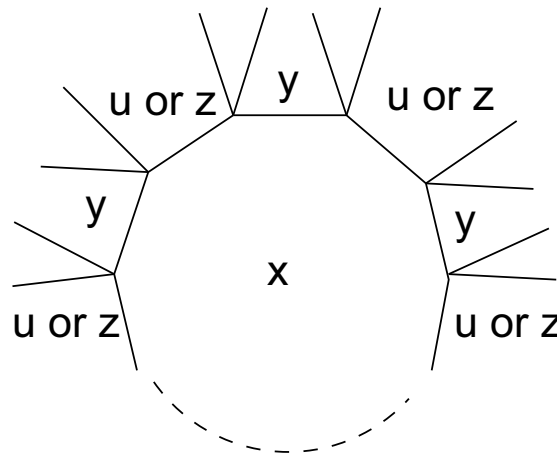
This rules out six vertex types $3 \cdot 7 \cdot 42$, $3 \cdot 8 \cdot 24$, $3 \cdot 9 \cdot 18$, $3 \cdot 10 \cdot 15$, $4 \cdot 5 \cdot 20$ and $5 \cdot 5 \cdot 10$.

Similarly: vertex type $x \cdot y \cdot u \cdot z$ is not possible when x is odd, $y \neq z$, and no three of the numbers are equal.

Reason: $x = y = z$ is not possible; let $x \neq z$.

- If $x \neq y$ consecutive edge-neighbors of x are y and z .
- If $x = y$ then $u \neq x, y$, and consecutive edge-neighbors of x are y and z , or $x = y$ and u .

In any case, edge neighbors $= y$ and $\neq y$ alternate.



This rules out four vertex types $3 \cdot 3 \cdot 4 \cdot 12$, $3 \cdot 3 \cdot 6 \cdot 6$, $3 \cdot 4 \cdot 3 \cdot 12$ and $3 \cdot 4 \cdot 4 \cdot 6$.

The only remaining types are the ones marked with “A”.

(ii) Uniqueness of the tiling for each vertex type.

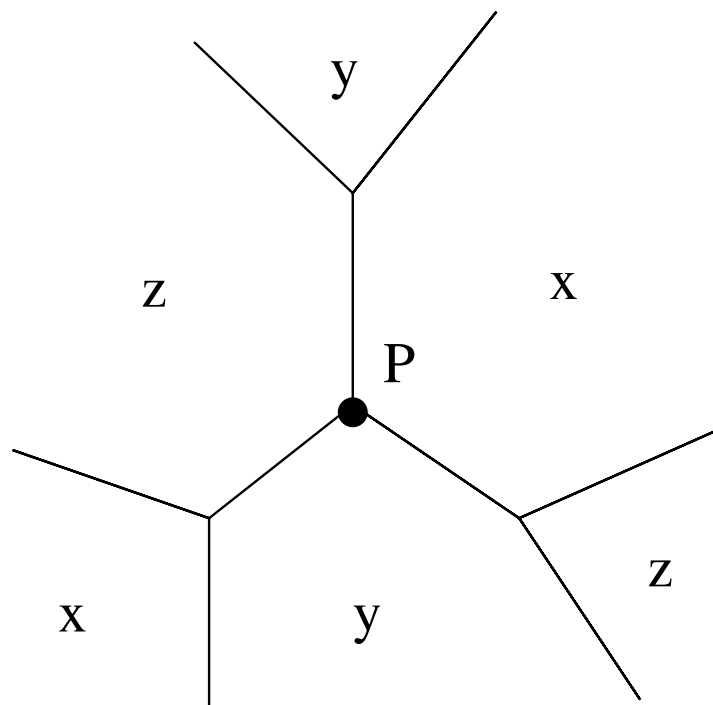
For each remaining vertex type there exists an archimedean tiling.

To prove uniqueness we show that, starting from known tiles around some vertex P of an archimedean tiling, the tiles around the neighboring vertices are uniquely forced.

Then (by induction on the distance of vertices from P) it follows that the tiles around all vertices are uniquely forced, so that the whole tiling is forced to be the same as the known archimedean tiling.

(The vertex type $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$ is an exception: there are two ways to complete the initial patch around P into an archimedean tiling. But these tilings are also isometric via an odd isometry.)

1) Vertex types $x \cdot y \cdot z$, that is, $3 \cdot 12 \cdot 12$, $4 \cdot 6 \cdot 12$, $4 \cdot 8 \cdot 8$, and $6 \cdot 6 \cdot 6$:

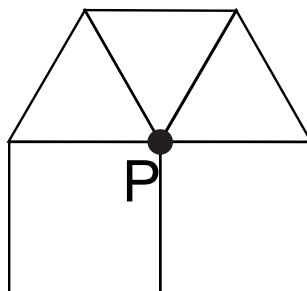


2) Vertex types $4 \cdot 4 \cdot 4 \cdot 4$ and $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$: Trivial because all tiles are congruent to each other.

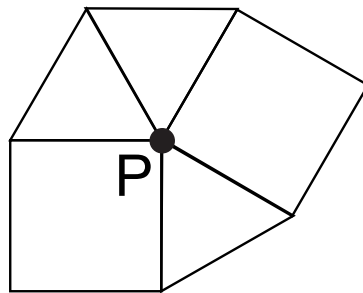
3) vertex types $x \cdot y \cdot x \cdot z$ where $y, z \neq x$, that is, types $3 \cdot 4 \cdot 6 \cdot 4$ and $3 \cdot 6 \cdot 3 \cdot 6$:

Opposite to x must be x .

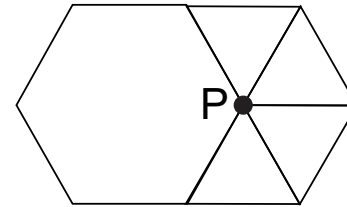
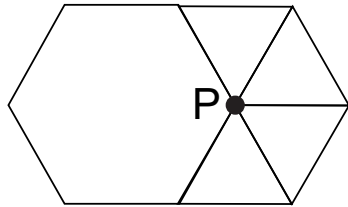
4) Vertex type $3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$:



5) Vertex type $3 \cdot 3 \cdot 4 \cdot 3 \cdot 4$:

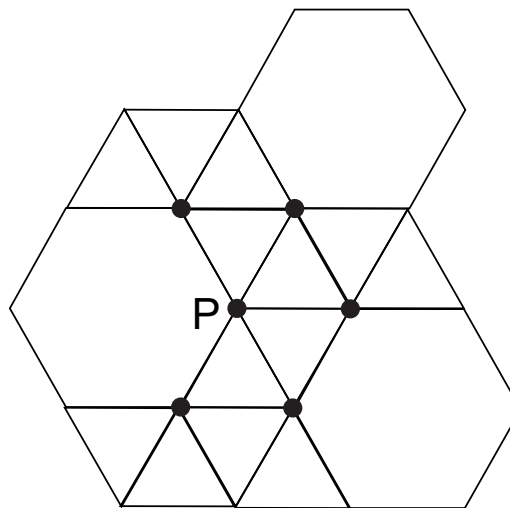


6) Vertex type $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$:



There are two ways of completing the polygons around the neighboring vertices. These ways are equivalent under an odd isometry.

Once the polygons around vertices of distance one from P are fixed, the rest of the tiling is uniquely forced:



Theorem. All archimedean tilings are **vertex-transitive**: For any two vertices P and Q there is a symmetry of the tiling that maps $P \mapsto Q$.

So we started with the weaker assumption that all vertices have the same type, and concluded the stronger property of vertex transitivity.

In fact: (**Except for $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$**) any isometry that maps the incident vertices around P onto the incident vertices around Q is a symmetry of the tiling. (In the case of $3 \cdot 3 \cdot 3 \cdot 3 \cdot 6$: there are two such isometries and one of them is a symmetry of the tiling.)