Tilings and Patterns: Homework 11 (1.12.2025)

1. Let $T = \{0, 1\}$. Determine for the given subset of $T^{\mathbb{Z}^2}$ whether it is a subshift, and whether it is a subshift of finite type.
   (a) $A_\tau = \{c \in T^{\mathbb{Z}^2} \mid \tau(c) = c\}$, where $\tau \in \mathbb{T}$ is an arbitrary fixed translation.
   (b) $B_k = \{c \in T^{\mathbb{Z}^2} \mid$ c contains exactly k occurrences of symbol 1 $\}$, where k is an arbitrary fixed positive integer,
   (c) $C_{\leq k} = \{c \in T^{\mathbb{Z}^2} \mid$ c contains at most k occurrences of symbol 1 $\}$, where k is an arbitrary fixed positive integer.

2. Let $T = \{0, 1\}$ and let $X \subseteq T^{\mathbb{Z}^2}$ be the set of configurations c such that every 2×2 pattern that appears in c has two 0's and two 1's.
   (a) Is X a subshift ? Is it a subshift of finite type ?
   (b) Prove that all elements of X are horizontally or vertically periodic (or both).
   (c) Is X transitive ?
   (d) Find some minimal subshift that is a subset of X.

3. Let $T = \{0, 1\}$ and let $X \subseteq T^{\mathbb{Z}^2}$ be the set of configurations c that consist of a (possibly empty) square of 1's on a background of 0's. More precisely, $c \in X$ if and only if there are $i, j \in \mathbb{Z}$ and $n \geq 0$ such that

$$c(x, y) = 1 \iff i \leq x < i + n \text{ and } j \leq y < j + n.$$

   (a) Is X a subshift ?
   (b) What are the elements in the closure $\overline{X}$?
   (c) Is the closure $\overline{X}$ a subshift ? Is it a subshift of finite type ?

4. Let P be a set of patterns over T such that $\Sigma(P) = \emptyset$. Prove that there is a finite $P' \subseteq P$ such that $\Sigma(P') = \emptyset$.

5. Let $X \subseteq T^{\mathbb{Z}^2}$ be clopen. Prove that

$$\bigcap_{\tau \in \mathbb{T}} \tau(X)$$

   is a subshift of finite type, and prove that every SFT is obtained in this way from some clopen X.

6. Prove that the following decision problems are undecidable concerning a given SFT Σ. (In the algorithmic questions, an SFT Σ is specified as a finite set P of forbidden patterns such that $\Sigma = \Sigma(P)$.)
   (a) Emptyness: Is $\Sigma = \emptyset$?
   (b) Finiteness: Is Σ finite ?
   (c) Transitivity: Is Σ transitive ?
   (d) Minimality: Is Σ minimal ?

7. The completion problem of a fixed subshift Σ is to determine if a given finite pattern is in $\text{Patt}(\Sigma)$. Prove that completion problem of any minimal SFT is decidable, so that every minimal SFT contains a recursive configuration. (Hint: The complement of the completion problem is semi-decidable for any SFT. For the other direction, use the fact that there is a semi-algorithm for the emptyness of a given SFT.)