Tilings and Patterns: Homework 8 (10.11.2025)

1. Determine if the Turing machine $M = (\{s, q, p, h\}, \{a, b\}, \delta, s, h, b)$ halts when started on the blank tape, where δ is given by

$$\begin{aligned}
\delta(s,b) &= (q,a,R)\\
\delta(s,a) &= (h,a,R)\\
\delta(q,b) &= (q,a,L)\\
\delta(q,a) &= (p,b,R)\\
\delta(p,b) &= (p,a,L)\\
\delta(p,a) &= (s,a,L)
\end{aligned}$$

   (Hint: Simulate the machine. Don't give up! But if you have done more than 50 moves you probably made a mistake. . .)

2. How many Wang tiles are in the set $\mathcal{P}_M$ in the Section 5.2 of the notes, when constructed from a Turing machine M with n states (one of which is the halting state) and m tape symbols? Give the number as a function of n and m. What is this number of tiles for the Turing machine defined in the Problem 1 above ?

3. Let A be a finite set of Wang prototiles that admits a valid tiling. Prove that A has a subset $B \subseteq A$ with the following property: B admits some valid tiling, and in every valid tiling with B every prototile in B is used infinitely many times.

4. Suppose a finite set of Wang tiles satisfies the following property: There exists a constant c such that, for every positive integer n, there are at most c valid tilings of the $n \times n$ square. Prove: the tile set only admits periodic tilings of the plane.

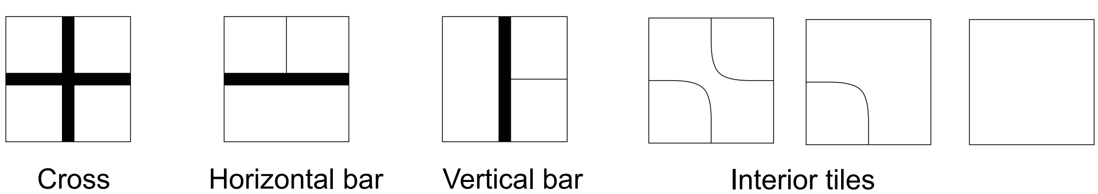
5. Using the construction in Lemma 5.5(ii), construct a set of Wang tiles that corresponds to the finite system (T, N, R) of allowed patterns where $T = \{0, 1\}$, $N = [(0, 0), (0, 1), (1, 0), (1, 1)]$ and the patterns

$$R = \{(a, b, c, d) \in T^4 \mid a + b + c + d = 1\}$$

   are allowed. (Every 2×2 block in a valid configuration must contain a single 1 and three 0's.)

6. Prove that the tile set you constructed in Problem 5 above only admits tilings that have the horizontal period $(2, 0)$ or the vertical period $(0, 2)$, or both.

7. Prove that the following six Wang tiles could be used instead of the tile set $\mathcal{Q}$ in the proof of Theorem 5.7 (Undecidability of the periodic tiling problem):



   (The matching rule is that lines must continue uninterrupted across tile boundaries. There are two different types of lines: thick and thin.) Thick lines specify *fault lines* as in the notes.

   More precisely, prove the following two facts:

   (a) For every $n \in \mathbb{Z}_+$ there is a valid, two-way periodic tiling where horizontal (and vertical) fault lines are spaced regularly at distance n from each other.

   (b) If a valid tiling contains at least two horizontal fault lines then it also must contain at least two vertical fault lines, and vice versa.