Tilings and Patterns: Homework 9 (17.11.2025)

1. Define, for any positive integer n,

$$f(n) = \max\{m \in \mathbb{N} \quad | \quad \begin{array}{l} \exists \text{ Wang tile set } P \text{ that does not admit a tiling of the} \\ \text{plane, } |P| \leq n \text{ and } P \text{ admits a tiling of an } m \times m \text{ square}\}. \end{array}$$

   Prove that $f(n)$ is a well defined number, and prove that there is no algorithm that on every input n returns a value that exceeds $f(n)$.

2. Let $\mathcal{P}$ be a given finite set of Wang prototiles, and let $B = (b, b, b, b) \in \mathcal{P}$ be a specified blank tile whose edges have the same blank color b. The tiling by copies of B is called the trivial tiling. A finite tiling is a valid tiling that contains only a finite number of tiles different from B. The *finite tiling problem* is the following decision problem: "Given $\mathcal{P}$ and B, does there exist a finite, non-trivial tiling of the plane?"

   Prove that the *finite tiling problem* is undecidable.

3. Determine if the following decision problems are decidable or undecidable: "Given a finite protoset P of Wang tiles and one specific prototile $t \in P$,...

   (a) ...does t belong to some valid strongly (=two-way) periodic tiling ?"

   (b) ...does t belong to every valid strongly (=two-way) periodic tiling ?"

4. For the two decision problems above [Problems 3(a) and (b)], determine if the problem is semi-decidable, and determine if the complement of the problem is semi-decidable.

5. Determine if the following decision problem is decidable or undecidable, and determine also whether the problem is semi-decidable: "Does a given finite set of Wang prototiles admit a tiling that is not strongly (=two-way) periodic ?"

6. Prove Theorem 5.11 of the notes: There exists a finite set of Wang prototiles whose completion problem is undecidable.

7. Show that if P admits a valid tiling and if the completion problem of P is decidable then P admits a recursive tiling.