Compactness of X could as well be defined using a dual concept:

Proposition. Topology of X is compact if and only if every family of closed sets whose intersection is empty has a finite subfamily whose intersection is empty.

Corollary. Let

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

be an infinite chain of closed sets in a compact space X. If $F_i \neq \emptyset$ for all i then

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

Compactness in metric spaces is equivalent to **sequential compactness**:

Proposition. Let (X, d) be a metric space. Set $A \subseteq X$ is compact if and only if every sequence of elements of A has a subsequence that converges to an element of A

(A **subsequence** of a sequence $x_1, x_2, \ldots$ is a sequence $x_{i_1}, x_{i_2}, \ldots$ for some $i_1 < i_2 < \ldots$.)

Proof

In compact metric spaces compact sets are exactly the closed sets:

Proposition A. Let X be a **compact** topological space. For $A \subseteq X$

$$A \text{ closed} \implies A \text{ compact}.$$

Proposition B. Let X be a **Hausdorff** topological space. For $A \subseteq X$

$$A \text{ compact} \implies A \text{ closed}.$$

Proofs.

Countability

Proposition. A compact metric space has a countable base and a countable dense set of points.

Proof.

Perfect sets

Recall that $x \in X$ is isolated if $\{x\}$ is open. On subsets $A \subset X$ we use the induced topology, so we have that $x \in A$ is **isolated in A** if $\{x\} = A \cap U$ for some open $U \subseteq X$.

A non-empty $A \subseteq X$ is **perfect** if it is closed and has no isolated points.

Proposition. Let (X, d) be a compact metric space. Then every perfect $A \subseteq X$ is uncountable.

Proof.