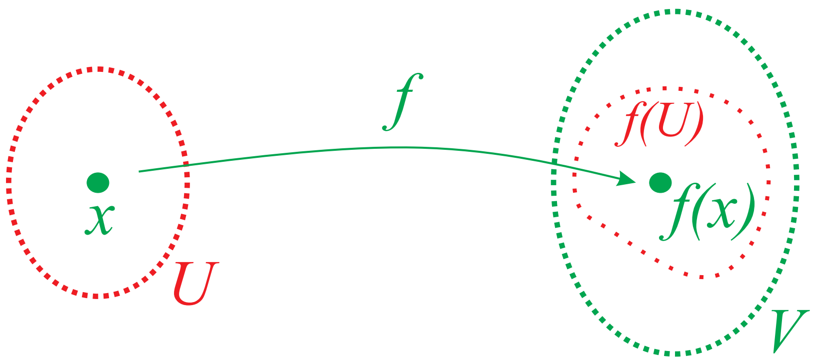
Continuity

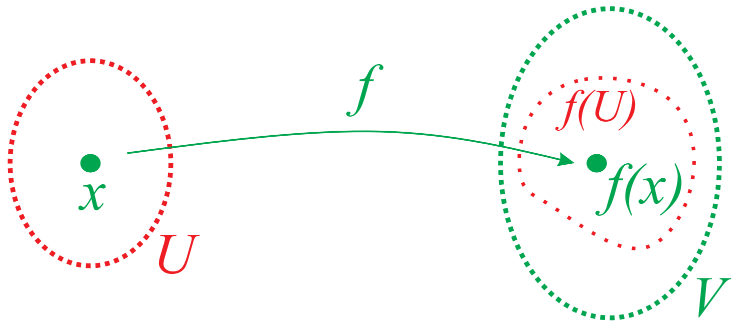
Let X, Y be topological spaces. Function $f : X \longrightarrow Y$ is **continuous at point** $x \in X$ if

$$V \subseteq Y \text{ open}, \ f(x) \in V \\ \implies \exists \text{ open } U \subseteq X : x \in U \text{ and } f(U) \subseteq V.$$

(For every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subseteq V$.



Function $f : X \longrightarrow Y$ is **continuous** if it is continuous at every $x \in X$.



Examples.

• If X has the discrete topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = \{x\}$.)

• If Y has the trivial topology the every $f : X \longrightarrow Y$ is continuous. (Choose $U = X$: works because $V = Y$.)

• A constant function ($\forall x \in X : f(x) = a$ for some fixed $a \in Y$) is continuous. (Choose $U = X$.)

• If X has the trivial topology and Y the discrete topology then constant functions are the only continuous functions:

Proposition. The following conditions are equivalent:

(i) Function $f : X \longrightarrow Y$ is continuous,

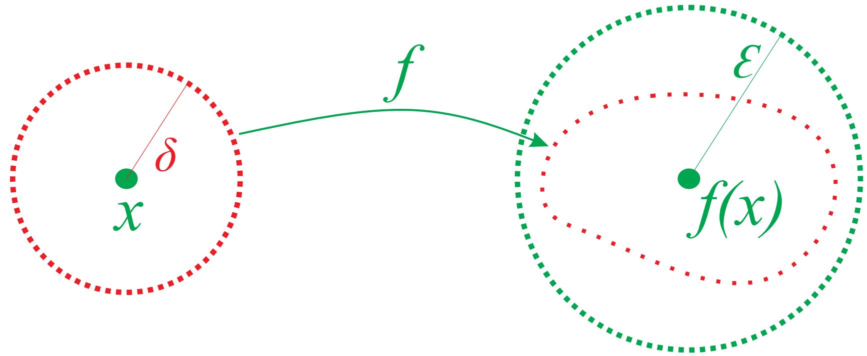
(ii) pre-image $f^{-1}(V)$ is open for each open $V \subseteq Y$,

(iii) pre-image $f^{-1}(F)$ is closed for each closed $F \subseteq Y$.

Proof.

In the metric case: f is continuous if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists \delta > 0\ :\ f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$



Proposition. Let $f : X \longrightarrow Y$ be continuous. For every compact A the set $f(A)$ is compact.

Proof.

Proposition. Let $f : X \longrightarrow Y$ be a continuous bijection where X is a compact and Y is a Hausdorff topological space. Then the inverse function $f^{-1} : Y \longrightarrow X$ is also continuous.

Proof.

If $f : X \longrightarrow Y$ is a bijection and both f and f^{-1} are continuous then f is a **homeomorphism** and spaces X and Y are **homeomorphic**. This is the "isomorphism" of topological structures.

Corollary. Continuous bijection between compact metric spaces is a homeomorphism.

Metric on $A^{\mathbb{Z}^2}$

The distance of configurations $c \neq e$ is

$$d(c,e) = 2^{-\min\{\|(i,j)\| \;\mid\; c(i,j) \neq e(i,j)\}}$$

where

$$\|(i,j)\| = \max\{|i|,|j|\}.$$

For any $r \in \mathbb{R}$ and $x, y \in A^{\mathbb{Z}^2}$ we have

$d(c,e) < 2^{-r} \iff e(i,j) = c(i,j)$ for all $|i|,|j| \leq r$.

The **open ball** of radius $\varepsilon = 2^{-r}$ centered at $c \in A^{\mathbb{Z}^2}$ is then

$B_\varepsilon(c) = \{e \in A^{\mathbb{Z}^2} \mid e(i,j) = c(i,j)$ for all $|i|,|j| \leq r\}$.

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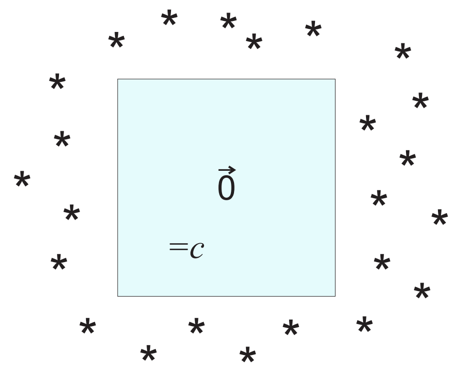
The ball consists of all configurations e that agree with c inside the square $D = \{-r, \ldots, r\} \times \{-r, \ldots, r\}$:



Thus open balls are precisely sets defined by finite patterns $p \in A^D$ for $D = \{-r, \ldots, r\} \times \{-r, \ldots, r\}$:

$$\{c \in A^{\mathbb{Z}^2} \mid c|_D = p\}.$$

Open balls:

$$\{c \in A^{\mathbb{Z}^2} \mid c|_D = p\}$$

for $p \in A^D$ with some $D = \{-r, \ldots, r\} \times \{-r, \ldots, r\}$.

Recall that open balls are a base of the topology: open sets are precisely the unions of open balls.

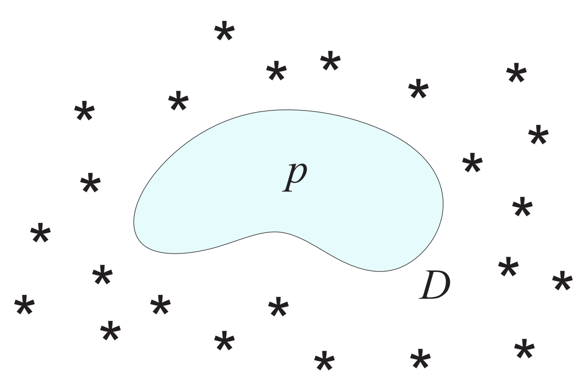
More generally we define a **cylinder** determined by a finite pattern $p \in A^D$ with any finite domain $D \subseteq \mathbb{Z}^2$ as the set of all configurations that have pattern p in domain D:

$$[p] = \{e \in A^{\mathbb{Z}^2} \mid e|_D = p\}.$$

We also denote this by

$$\mathrm{Cyl}(c, D)$$

for any $c \in [p]$.



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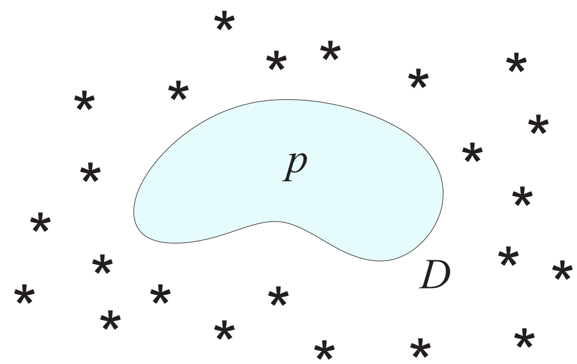

- Open balls are cylinders (where D is a square centered at $\vec{0}$),
- Every cylinder is a union of open balls: Indeed, if $D \subseteq E$ and $p \in A^D$ then

$$[p] = \bigcup_{\substack{q \in A^E \\ q|_D = p}} [q].$$

Thus cylinders form a **basis** of the topology.

Equivalently: A set $U \subseteq A^{\mathbb{Z}^2}$ is open iff for every $c \in U$ there exists a finite $D \subseteq \mathbb{Z}^d$ such that $[c|_D] \subseteq U$.

Cylinders are also closed because their complements are open:

$$A^{\mathbb{Z}^d} \setminus [p] = \bigcup_{\substack{q \in A^D \\ q \neq p}} [q].$$

Thus cylinders are a clopen basis of the topology.

Remark. Our **metric** is **not translation invariant**: difference at the center cell makes two configurations more distant from each other than a difference at a cell far from the center.

However, the **topology** that the metric induces is **translation invariant**: Translations of cylinders are cylinders so the base (=the set of cylinders) is invariant under translations. In the topology the center cell is not more important than any other cell.

Another remark. The same topology (homeomorphic to the **Cantor space**) is induced by many other metrics. If the open balls are cylinders, and if all cylinders are subsets of some open balls then the metric defines the same topology.

For example, we can define a metric by

$$d(c,e) = f(\min\{g(i,j) \mid c(i,j) \neq e(i,j)\})$$

where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is any decreasing function such that

$$\lim_{n\to\infty} f(n) = 0,$$

and the function $g : \mathbb{Z}^2 \longrightarrow \mathbb{R}_+$ can be any function such that for every a there are only finitely many $(i,j) \in \mathbb{Z}^2$ with the property $g(i,j) < a$.

This always defines the same topology as our choice

$$f(x) = 2^{-x}, \qquad g(i,j) = \max\{|i|, |j|\}.$$

For example,

$$d(c,e) = \frac{1}{1 + \min\{|i| + |j| \mid c(i,j) \neq e(i,j)\}}$$

could be used as the metric.

Yet another remark.

- The set of strongly periodic configurations is a countable dense subset of $A^{\mathbb{Z}^d}$.
- The set of cylinders is a countable base of the topology.

Proof.

Earlier: we said that a sequence $c_1, c_2, \ldots$ of configurations **converges** to $c \in A^{\mathbb{Z}^2}$ if

$$\forall (i,j) \in \mathbb{Z}^2,\ \exists N \in \mathbb{N},\ \forall k > N \ :\ c_k(i,j) = c(i,j).$$

This concept of convergence is identical to convergence in our metric:

$\Longleftarrow$ Suppose $c_1, c_2, \ldots$ converges to c under the metric.

$\Longrightarrow$ Suppose $c_1, c_2, \ldots$ converges to c in the earlier sense.

We have proved that every sequence has a converging subsequence.

We have proved that sequential compactness implies compactness (when there is a countable base).

Theorem. The metric space $A^{\mathbb{Z}^2}$ is compact.