Let $\vec{n} \in \mathbb{Z}^2$.

The **translation**

$$\tau_{\vec{n}} : A^{\mathbb{Z}^2} \longrightarrow A^{\mathbb{Z}^2}$$

by the vector $\vec{n}$ shifts the configurations by $\vec{n}$: It maps $c \mapsto e$ where

$$\forall \vec{m} \in \mathbb{Z}^2 \;:\; e(\vec{m}) = c(\vec{m} - \vec{n}).$$

Translations are **bijective**, and $\tau_{\vec{n}}$ and $\tau_{-\vec{n}}$ are inverses of each other. All translations **commute** with each other.

The **east shift** σ_e and the **north shift** σ_n are translations by vectors $(1, 0)$ and $(0, 1)$ respectively. They generate all translations: For all $i, j \in \mathbb{Z}$,

$$\tau_{i,j} = \sigma_e^i \sigma_n^j.$$

We denote by $\mathbb{T}$ the **set of all translations**.

Any translation applied to a cylinder gives a cylinder:

$$\tau_{\vec{n}}\left(\mathrm{Cyl}(c, D)\right) = \mathrm{Cyl}(\tau_{\vec{n}}(c), D + \vec{n}).$$

This means that translations $\tau_{\vec{n}}$ are **continuous**.

So we have a compact metric space $A^{\mathbb{Z}^2}$, equipped with continuous transformations in $\mathbb{T}$. This is the set-up studied in topological dynamics. The system $(A^{\mathbb{Z}^2}, \mathbb{T})$ is a **dynamical system**.

Subshifts

A set $\Sigma \subseteq A^{\mathbb{Z}^2}$ is **translation invariant** if

$$\tau(\Sigma) = \Sigma$$

for every $\tau \in \mathbb{T}$. It is enough to verify that $\sigma_e(\Sigma) = \Sigma$ and $\sigma_n(\Sigma) = \Sigma$.

A topologically closed, translation invariant set is a (two-dimensional) **subshift**. The entire configuration space $A^{\mathbb{Z}^2}$ is also called the (two-dimensional) **full shift** over A.

The system $(\Sigma, \mathbb{T})$ is also a dynamical system, a **subsystem** of $(A^{\mathbb{Z}^2}, \mathbb{T})$.

A finite **pattern** over A is a pair (D, p) where $D \subseteq \mathbb{Z}^2$ is finite, the domain of the pattern, and $p : D \longrightarrow A$.

(We often omit D from the notation and talk about a pattern $p \in A^D$, the domain D being implicitly assumed.)

Let us denote by $P(A)$ the set of **all finite patterns** over A. Clearly $P(A)$ is countable as the number of finite subsets of $\mathbb{Z}^2$ is countable.

Pattern $p \in A^D$ is a **subpattern** of a configuration $c \in A^{\mathbb{Z}^2}$ if $c|_D = p$.

Configurations that have p as a subpattern form the cylinder

$$[p] = \{c \in A^{\mathbb{Z}^2} \mid c|_D = p\}.$$

This is of course the same cylinder as $\mathrm{Cyl}(c, D)$ for any configuration c in the cylinder.

We say that the pattern $p \in A^D$ **appears** in c if p is a subpattern of $\tau(c)$ for some translation $\tau \in \mathbb{T}$:

For any configuration c let **Patt**(c) be the set of all finite patterns that appear in c:

$$\text{Patt}(c) = \{p \in A^D \mid \exists \tau \in \mathbb{T} \ : \ \tau(c)|_D = p\}.$$

For any set $S \subseteq A^{\mathbb{Z}^2}$ of configurations we denote

$$\text{Patt}(S) = \bigcup_{c \in S} \text{Patt}(c)$$

for the set of finite patterns that appear in some elements of S.

For any set P of finite patterns we define the set

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \text{Patt}(c) \cap P = \emptyset\}$$

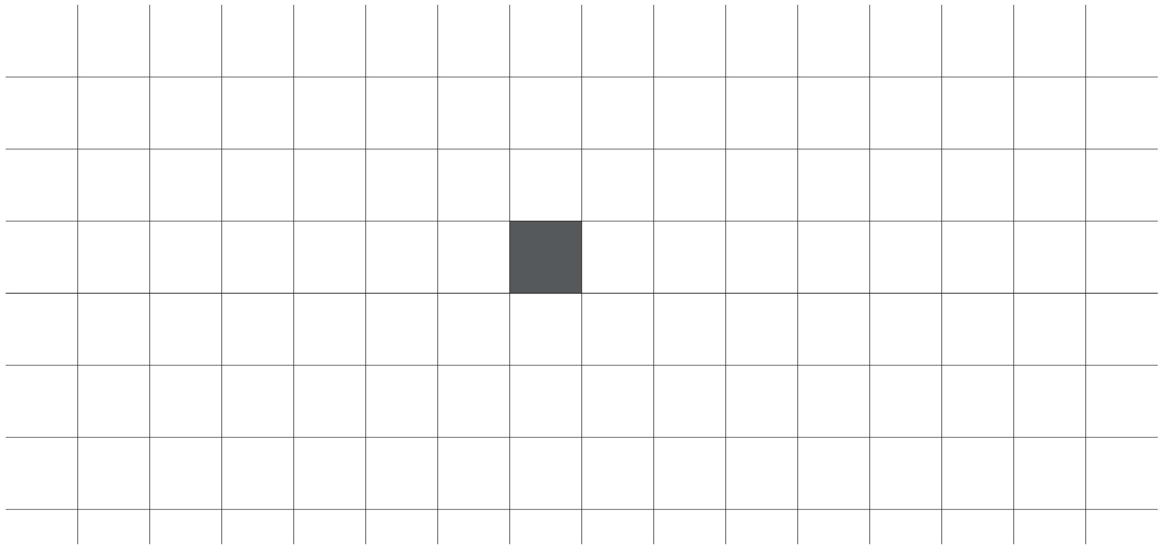
of configurations in which no element of P appears.

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \mathrm{Patt}(c) \cap P = \emptyset\}$$

Theorem. Σ is a subshift if and only if $\Sigma = \Sigma(P)$ for some set P of finite patterns.

Proof.

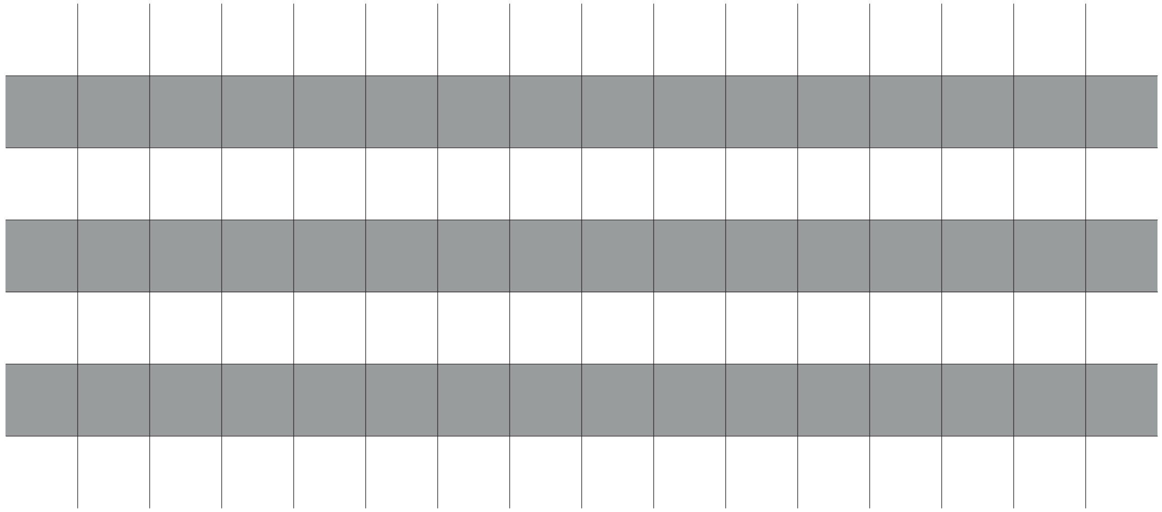
Example. The set consisting of the configrations with a single black cell on a white background, and the all white configuration:



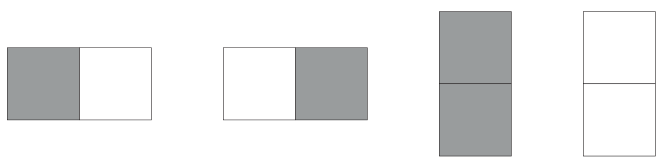
This is translation invariant and topologically closed. So it is a subshift (the **sunny-side-up** subshift).

It is defined by forbidding all patterns $p \in \{0,1\}^D$ where $|D| = 2$ and $p(i,j) = 1$ for both $(i,j) \in D$.

Example. The set of the two configurations where black and white horizontal rows alternate:

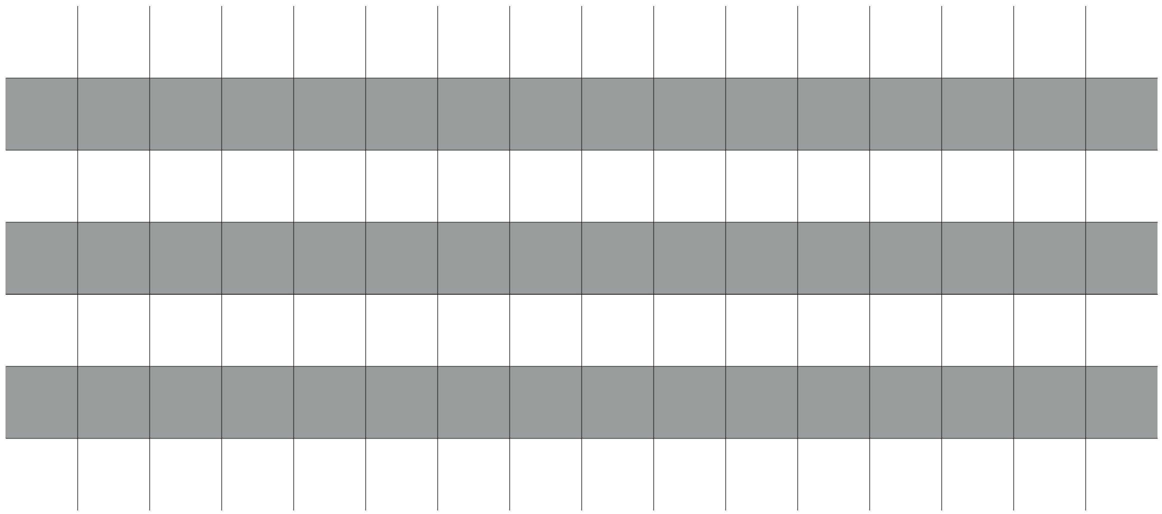


This is the subshift $\Sigma(P)$ determined by forbidding the patterns

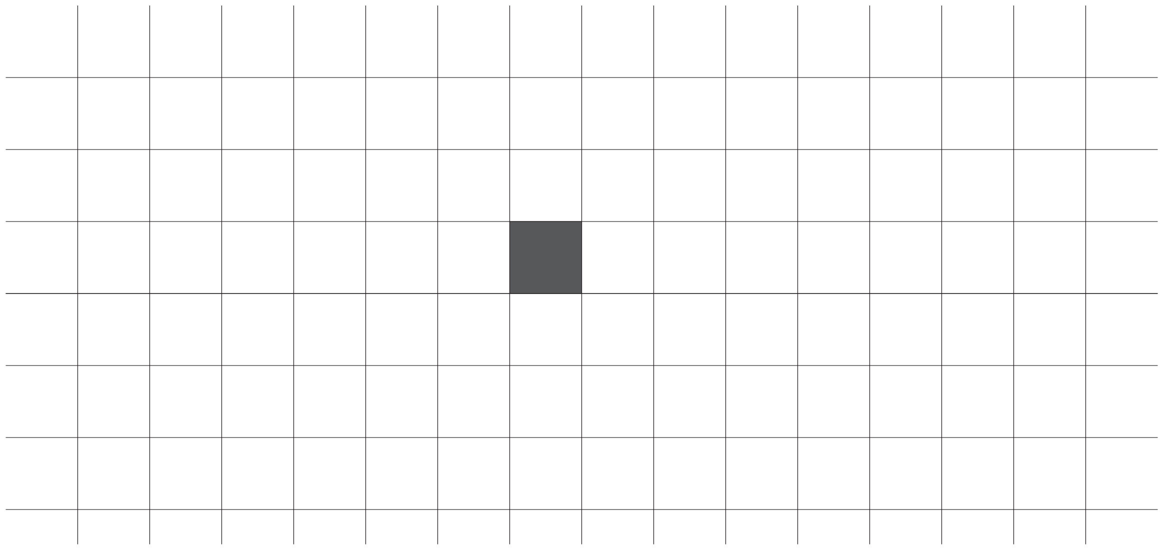


Subshifts $\Sigma(P)$ for finite P are called **subshifts of finite type (SFT)**. So a SFT can be specified by giving a finite collection P of forbidden patterns.

Example. The alternating stripes subshift is an SFT.



Example. The sunny-side-up subshift is **not** an SFT.



The set $V(T)$ of valid tilings by a Wang tile set T is an SFT.

Conversely, we have seen a method of turning any SFT Σ into an "equivalent" set T of Wang tiles. In this construction the Wang tiling corresponding to a configuration $c \in \Sigma$ is obtained by **sliding an $n \times n$ window** across c and reading the patterns (which are the tiles) inside the window.

The “sliding window” function

$$h : \Sigma \longrightarrow V(T)$$

is a continuous bijection. It also commutes with all translations:

$$h \circ \tau = \tau \circ h$$

for all $\tau \in \mathbb{T}$.

A translation commuting continuous bijection is called a **conjugacy** between the dynamical systems $(\Sigma, \mathbb{T})$ and $(V(T), \mathbb{T})$. Conjugacies are the isomorphisms between dynamical systems. Conjugate dynamical systems are “the same”.

Orbit closure

For any $c \in A^{\mathbb{Z}^2}$ the set

$$\mathcal{O}(c) = \{\tau(c) \mid \tau \in \mathbb{T}\}$$

is the **orbit** of c. The set $\mathcal{O}(c)$ is translation invariant.

The **orbit closure**

$$\overline{\mathcal{O}(c)}$$

of c is the topological closure of the orbit.

Lemma. The orbit closure $\overline{\mathcal{O}(c)}$ is a subshift, for every $c \in A^{\mathbb{Z}^2}$.

Proof.

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Lemma. The orbit closure $\overline{\mathcal{O}(c)}$ is a subshift, for every $c \in A^{\mathbb{Z}^2}$.

Proof. We only need to prove that the orbit closure is translation invariant:

Let $e \in \overline{\mathcal{O}(c)}$ and $\tau \in \mathbb{T}$. For any open neighborhood U of $\tau(e)$, the open set $\tau^{-1}(U)$ is an open neighborhood of e. Hence $\tau^{-1}(U) \cap \mathcal{O}(c) \neq \emptyset$, so that $U \cap \mathcal{O}(c) \neq \emptyset$. This means that $\tau(e) \in \overline{\mathcal{O}(c)}$.

The orbit closure $\overline{\mathcal{O}(c)}$ is the subshift **generated** by c: it is the intersection of all subshifts that contain c.

Indeed, if Σ is a subshift and $c \in \Sigma$ then

$\mathcal{O}(c) \subseteq \Sigma$ (because Σ is translation invariant) and

$\overline{\mathcal{O}(c)} \subseteq \Sigma$ (because Σ is closed).

Remark: If $P_1 \subseteq P_2$ then $\Sigma(P_2) \subseteq \Sigma(P_1)$.

Follows directly from the definition

$$\Sigma(P) = \{c \in A^{\mathbb{Z}^2} \mid \text{Patt}(c) \cap P = \emptyset\}$$

Let P be the complement of $\text{Patt}(c)$, that is, the set of all finite patters that do not appear in c:

$$P = \{p \mid p \notin \text{Patt}(c)\}.$$

Then $\overline{\mathcal{O}(c)} = \Sigma(P)$.

Proof.

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Then $\overline{\mathcal{O}(c)} = \Sigma(P)$.

Proof. $\overline{\mathcal{O}(c)} = \Sigma(P')$ for some P'.

- As $\Sigma(P)$ is a subshift containing c, we have $\overline{\mathcal{O}(c)} \subseteq \Sigma(P)$.
- As $P' \subseteq P$, by the remark, $\Sigma(P) \subseteq \Sigma(P') = \overline{\mathcal{O}(c)}$.

$$\overline{\mathcal{O}(c)} = \Sigma(P) \text{ for } P = \{p \mid p \notin \mathrm{Patt}(c)\}$$

Lemma. $e \in \overline{\mathcal{O}(c)}$ if and only if $\mathrm{Patt}(e) \subseteq \mathrm{Patt}(c)$.

Proof.

$$\overline{\mathcal{O}(c)} = \Sigma(P) \text{ for } P = \{p \mid p \notin \mathrm{Patt}(c)\}$$

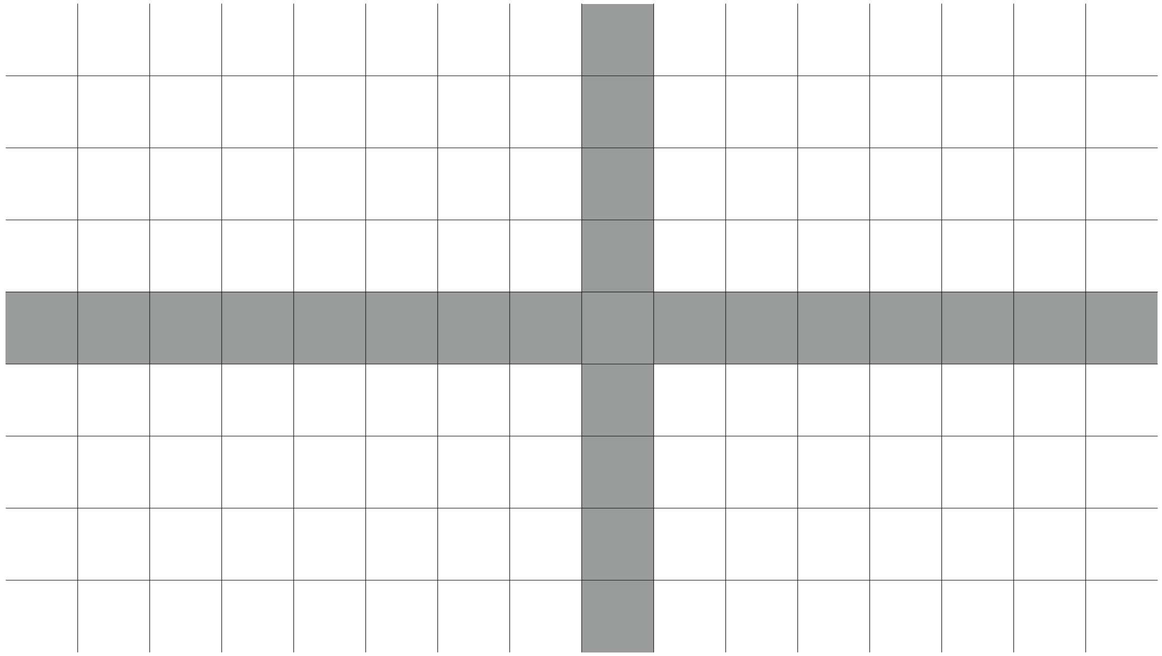
Lemma. $e \in \overline{\mathcal{O}(c)}$ if and only if $\mathrm{Patt}(e) \subseteq \mathrm{Patt}(c)$.

Proof.

$$e \in \overline{\mathcal{O}(c)} = \Sigma(P) \iff \mathrm{Patt}(e) \cap P = \emptyset \iff \mathrm{Patt}(e) \subseteq \mathrm{Patt}(c).$$

Example. Let $A = \{0, 1\}$ and let $c \in A^{\mathbb{Z}^2}$ be the infinite cross:

- $c(i, 0) = c(0, i) = 1$ for all $i \in \mathbb{Z}$, and
- $c(i, j) = 0$ if $i, j \neq 0$.



The orbit closure $\overline{\mathcal{O}(c)}$ contains the following configurations:

Transitivity

A non-empty subshift Σ is called **transitive** if for every $p_1, p_2 \in \text{Patt}(\Sigma)$ there exists $c \in \Sigma$ such that

$$p_1, p_2 \in \text{Patt}(c).$$

In other words, any two patterns that appear in some elements of Σ, appear in the same element of Σ.

Transitive subshifts are exactly the orbit closures of configurations:

Theorem. Subshift Σ is transitive if and only if

$$\Sigma = \overline{\mathcal{O}(c)}$$

for some $c \in \Sigma$.

Proof.

An element $c \in \Sigma$ is **transitive** in Σ if $\Sigma = \overline{\mathcal{O}(c)}$.

In other words, a transitive configuration contains all finite patterns that appear in any configuration of the subshift.

Corollary. A subshift has a transitive element if and only if the subshift is transitive.

Example. The infinite cross is transitive in its orbit closure. The orbit closure contains a non-transitive subset

$$\Sigma = \overline{\mathcal{O}(c_v)} \cup \overline{\mathcal{O}(c_h)}$$

generated by the horizontal and vertical black rows.