Minimality

A non-empty subshift Σ is called **minimal** if the only subshifts contained in Σ are $\emptyset$ and Σ.

By the following theorem Σ is minimal if and only if the orbits of all its elements are dense in Σ:

Theorem. Let Σ be a non-empty subshift. The following are equivalent:

(i) Σ is minimal.

(ii) All elements of Σ are transitive in Σ.

(iii) $\mathrm{Patt}(e) = \mathrm{Patt}(c)$ for all $e, c \in \Sigma$.

Proof.

Theorem. Every non-empty subshift Σ has a subset that is a minimal subshift.

Proof.

Periodicity and recurrence properties

Recall: A configuration $c \in A^{\mathbb{Z}^2}$ is (one-way) **periodic** if there exists $\vec{n} \in \mathbb{Z}^2 \setminus \vec{0}$ such that $c = \tau_{\vec{n}}(c)$.

It is **strongly** (or two-way) **periodic** if it is periodic with two linearly independent periods $\vec{n}_1$ and $\vec{n}_2$. A strongly periodic configuration is always periodic with horizontal and vertical periods $(0, n)$ and $(n, 0)$ for some $n > 0$.

If a **subshift of finite type** contains a one-way periodic element then it contains a strongly periodic element as well.

(This was show for Wang tilings. All SFT are conjugate to Wang tilings. Conjugacy preserves periods.)

Remark. The orbit $\mathcal{O}(c)$ of c is finite if and only if c is strongly periodic. Also, the orbit is closed if and only if c is strongly periodic (homework).

Uniform recurrence

A configuration $c \in A^{\mathbb{Z}^2}$ is **uniformly recurrent** if for every finite pattern $p \in \mathrm{Patt}(c)$ there exists n such that p appears in c inside every $n \times n$ square.

More precisely: c is uniformly recurrent iff

$$
\begin{aligned}
&(\forall p \in \mathrm{Patt}(c)) \\
&\qquad (\exists_{\text{finite}}\, \mathbb{T}' \subseteq \mathbb{T}) \\
&\qquad\qquad (\forall \tau \in \mathbb{T}) \\
&\qquad\qquad\qquad (\exists \tau' \in \mathbb{T}') \\
&\qquad\qquad\qquad\qquad \tau'(\tau(c)) \in [p].
\end{aligned}
$$

Uniformly recurrent configurations generate minimal subshifts:

Theorem. Subshift $\overline{\mathcal{O}(c)}$ is minimal if and only if c is uniformly recurrent.

Proof.