Thus all configurations of a minimal subshift are uniformly recurrent.

There are two possibilities in this case:

- either all of them are strongly periodic, in which case the subshift is their finite orbit,
- or all elements are non-periodic (but uniformly recurrent) configurations that contain exactly the same finite patterns.

In the second case the subshift contains an uncountable number of elements:

Theorem. A minimal subshift is either finite or uncountably infinite.

Proof.

A summary, in terms of Wang tilings:

Corollary. If a tile set admits a valid tiling then it admits a uniformly recurrent tiling. If it admits a uniformly recurrent tiling that is not strongly periodic, then it admits uncountably many different tilings. In particular, every aperiodic tile set admits uncountably many valid, uniformly recurrent tilings.

A weaker recurrence property

A configuration is **recurrent** if every finite pattern that appears in c appears more than once in c.

More precisely: c is recurrent iff for all finite patterns p:

$$(\forall p \in \mathrm{Patt}(c))\ (\exists \vec{n} \neq \vec{m}) \qquad \tau_{\vec{n}}(c), \tau_{\vec{m}}(c) \in [p].$$

Then, in fact, patterns that appear in c appear infinitely many times in c:

Lemma. Configuration c is recurrent if and only if for every $p \in \mathrm{Patt}(c)$ there are infinitely many translations $\tau \in \mathbb{T}$ such that $\tau(c) \in [p]$.

Proof.

Two stronger recurrence properties

Configuration c is **quasi-periodic** if every occurrence of a finite pattern in c is part of a two-way periodic repetition of the pattern (but the period may be different for different patterns).

More precisely: for every finite $D \subseteq \mathbb{Z}^2$ there exist linearly independent $\vec{a}, \vec{b} \in \mathbb{Z}^2$ such that

$$(\forall i, j \in \mathbb{Z}) \qquad \tau_{i\vec{a}+j\vec{b}}(c)|_D = c|_D.$$

Configuration c is **isochronous** if every finite pattern of c appears two-way periodically in c.

More precisely: for every finite $D \subseteq \mathbb{Z}^2$ there exists an offset vector $\vec{c} \in \mathbb{Z}^2$ and two linearly independent $\vec{a}, \vec{b} \in \mathbb{Z}^2$ such that

$$(\forall i, j \in \mathbb{Z}) \qquad \tau_{i\vec{a}+j\vec{b}+\vec{c}}(c)|_D = c|_D.$$

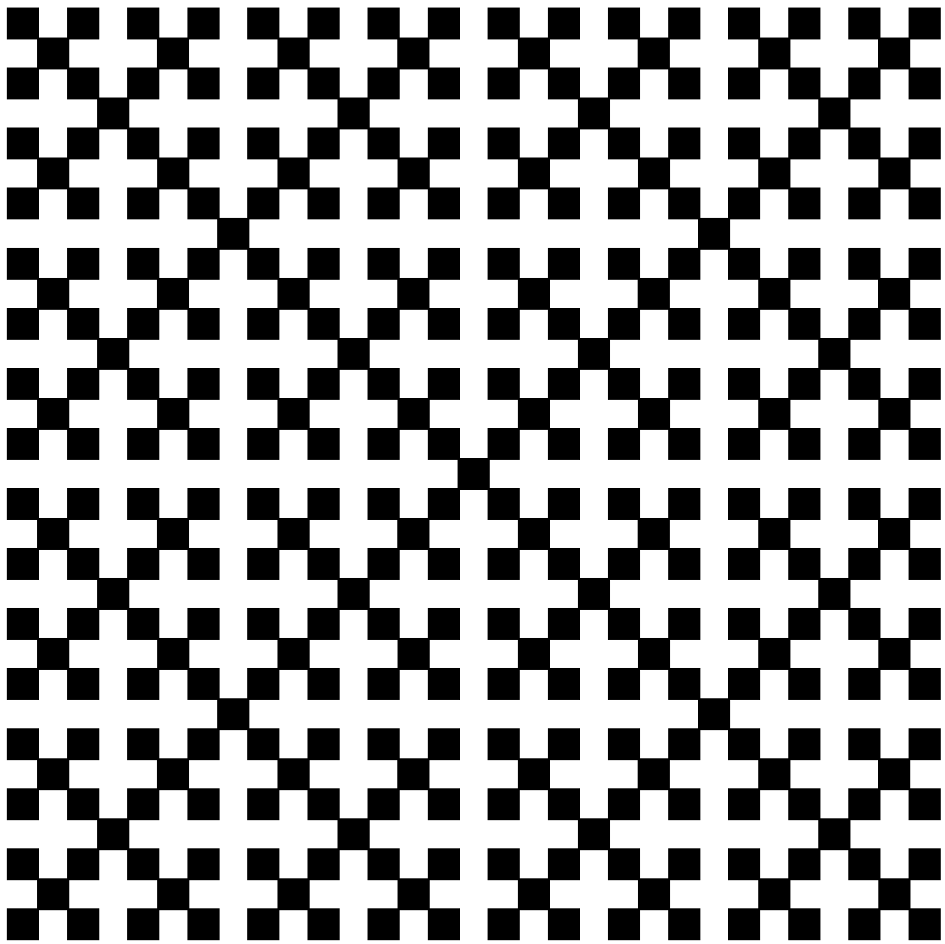
The following implications are obvious from the definitions:

$$\text{strongly periodic} \Longrightarrow \text{quasi-periodic} \Longrightarrow \text{isochronous}$$
$$\Longrightarrow \text{uniformly recurrent} \Longrightarrow \text{recurrent}$$

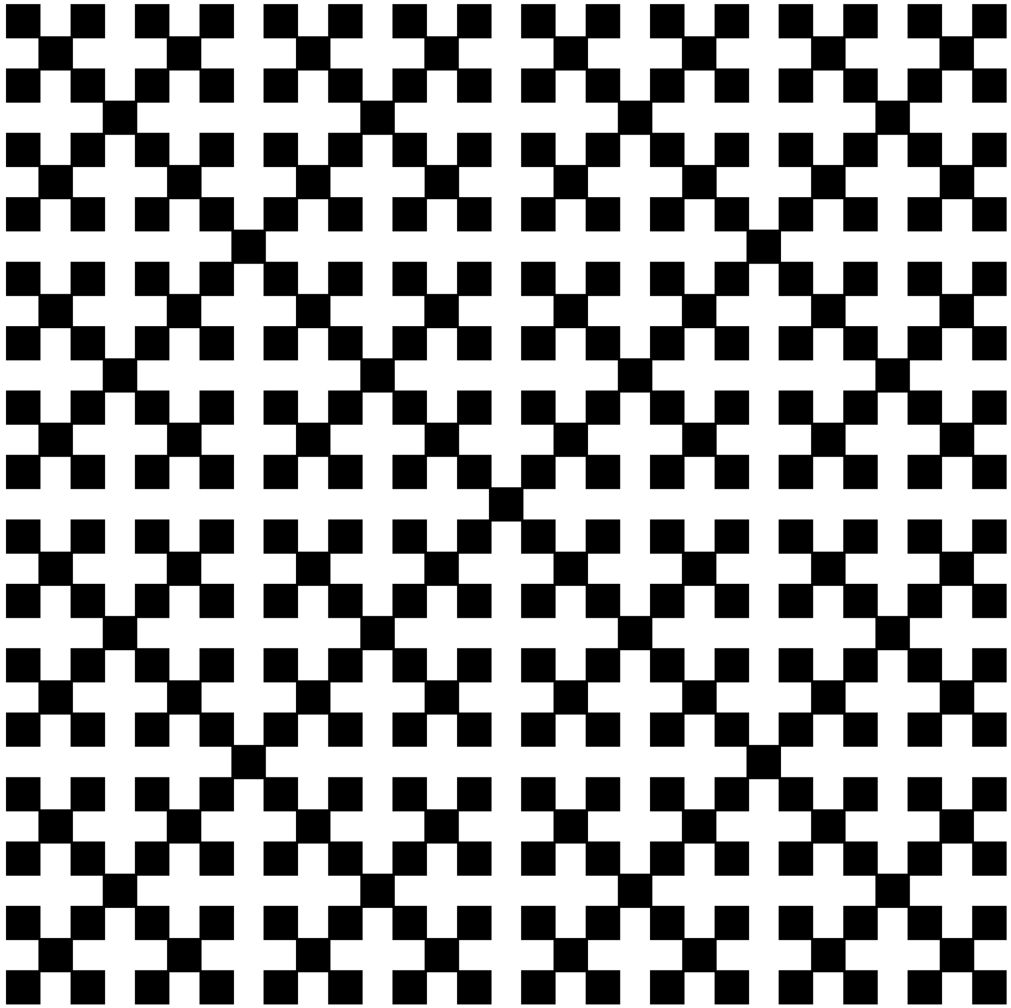
Example. Let $\deg_2(n)$ denote the largest power of 2 that divides integer n. (And set $\deg_2(0) = \infty$.)

Define configuration $c \in \{0,1\}^{\mathbb{Z}^2}$ as follows:

$$c(i,j) = 1 \text{ if and only if } \deg_2(i) = \deg_2(j).$$



$$c(i, j) = 1 \text{ if and only if } \deg_2(i) = \deg_2(j).$$



(Black squares are as the crosses in a tiling by Robinson's tile set.)
This configuration is **isochronous** but **not quasi-periodic**.

Isolated points

Configuration $c \in \Sigma$ is **isolated** in Σ if for some finite pattern p

$$[p] \cap \Sigma = \{c\}.$$

Lemma. Let Σ be a subshift. All $c \in \Sigma$ are isolated in Σ if and only if Σ is finite.

Proof.

Finite subshifts are exactly the ones whose elements are all two-way periodic:

Theorem. A subshift Σ is finite if and only if every $c \in \Sigma$ is two-way periodic.

Proof.