Sensitivity concepts from topolgical dynamics

Configuration $c \in \Sigma$ is an **equicontinuity point** if

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall e \in \Sigma)(\forall \tau \in \mathbb{T})$

$$d(c, e) < \delta \Longrightarrow d(\tau(c), \tau(e)) < \varepsilon.$$

In other words, if e is chosen sufficiently close to c then all translates $\tau(e)$ and $\tau(c)$ are close to each other.

Lemma. c is an equicontinuity point in Σ if and only if it is isolated in Σ.

Proof.

The subshift Σ is called **equicontinuous** if all $c \in \Sigma$ are equicontinuity points.

Corollary. A subshift is equicontinuous if and only if it is finite.

A subshift Σ is called **sensitive** if there exists $\varepsilon > 0$, called the **sensitivity constant**, such that

$$(\forall c \in \Sigma)(\forall \delta > 0)(\exists e \in \Sigma)(\exists \tau \in \mathbb{T})$$

$$d(c, e) < \delta \text{ and } d(\tau(c), \tau(e)) > \varepsilon.$$

In other words, arbitrarily close to each c there is another configuration e such that for a suitable translation τ the configurations $\tau(c)$ and $\tau(e)$ are not close to each other.

Lemma. Σ is sensitive if and only if it has no isolated points.

Proof.

A subshift is **expansive** if

$$(\exists \varepsilon > 0)(\forall c, e \in \Sigma)$$

$$c \neq e \implies (\exists \tau \in \mathbb{T})\ d(\tau(c), \tau(e)) > \varepsilon.$$

Lemma. All subshifts are expansive.

Proof.

Theorem. Let Σ be a subshift.

(i) Σ is expansive.

(ii) Σ is sensitive if and only if it has no isolated points.

(ii) Σ is equicontinuous if and only all its elements are isolated, i.e., the subshift is finite.

Aperiodic geometric (polygonal) tile sets

Recall: Any Wang tile set can be converted into prototiles that are polygons (bumps and dents construction).

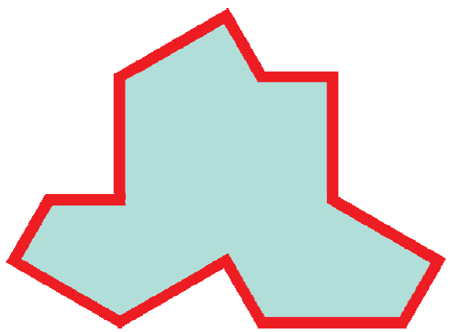
If the Wang tile set is aperiodic then the corresponding polygons admit a tiling but do not admit a periodic tiling (=tiling with a translational symmetry). We say the geometric tile set is also **aperiodic**.

In the exercises we had an example of an aperiodic set of six tiles, similar to Robinson's Wang tile set.

Theorem. There exists a protoset of polygons that admits a valid tiling but does not admit a valid tiling with a non-trivial symmetry.

The smallest aperiodic Wang protoset contains 11 tiles, but with geometric tiles a single tile is enough to force non-periodicity!

A polygonal prototile (the **hat**) was recently discovered that is alone aperiodic: there exist monohedral tilings of $\mathbb{R}^2$ using the hat but none of these tilings has a translational symmetry.



We discuss the hat in more details later in the class.

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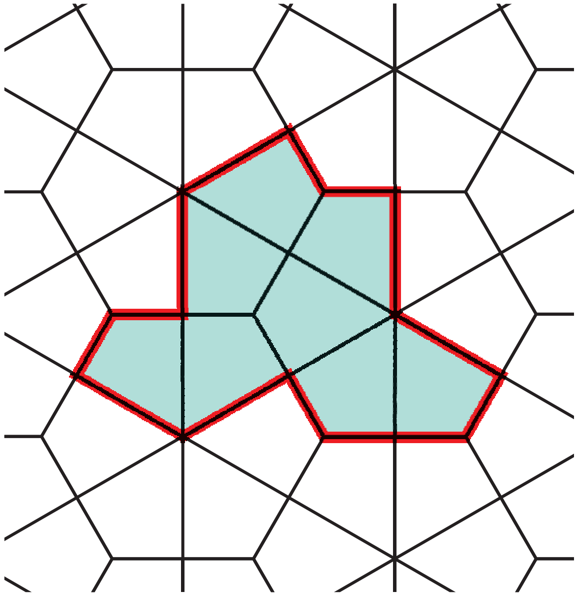

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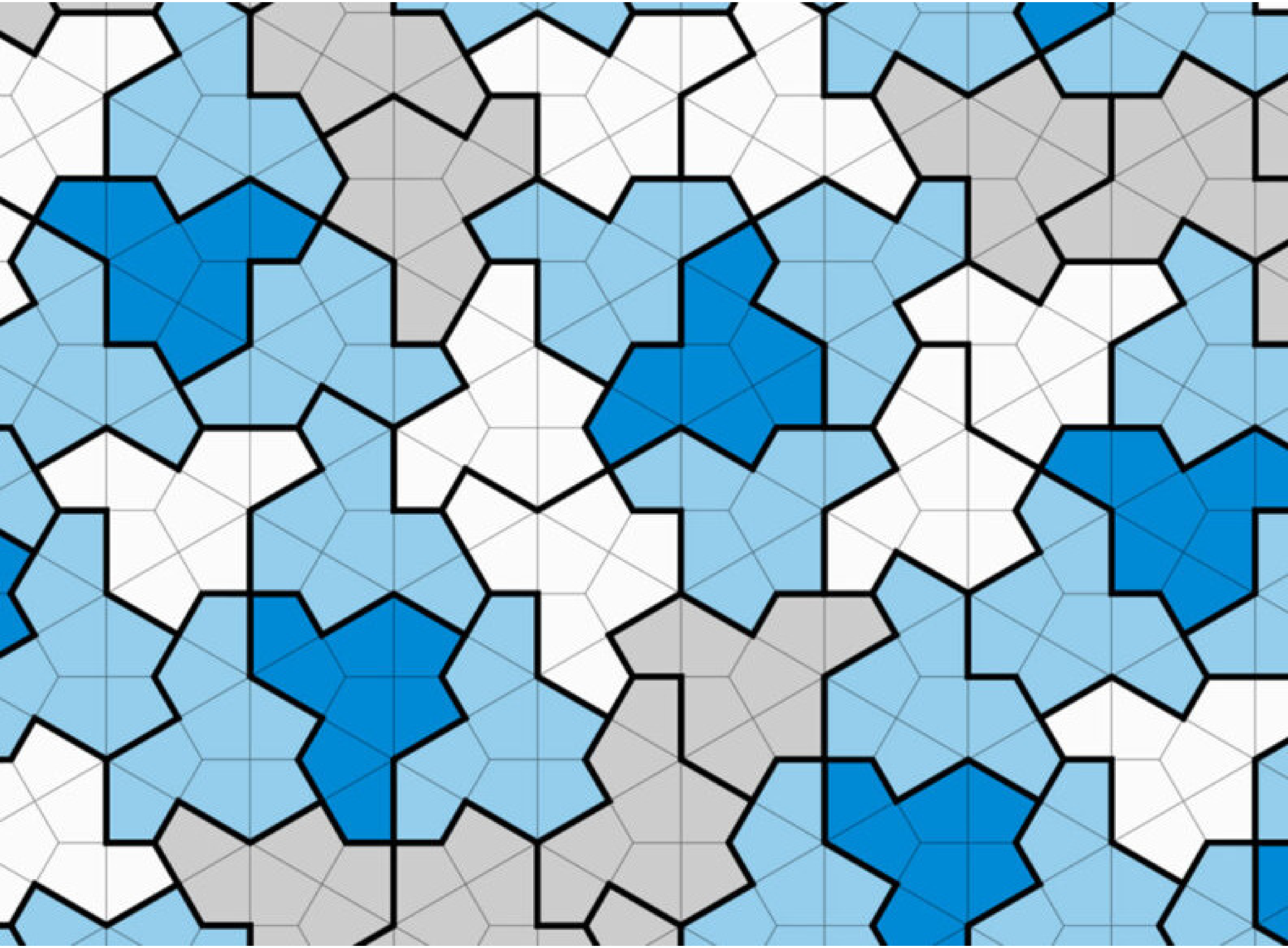
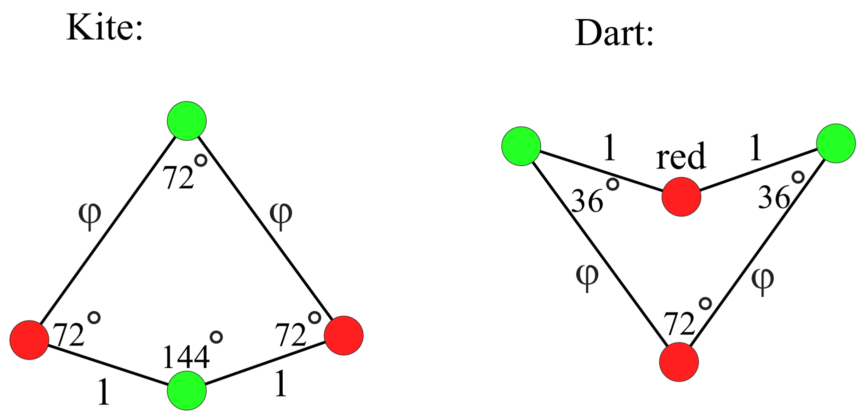


In 1974 Roger Penrose presented a famous aperiodic pair of polygonal tiles: the **kite** and the **dart**.



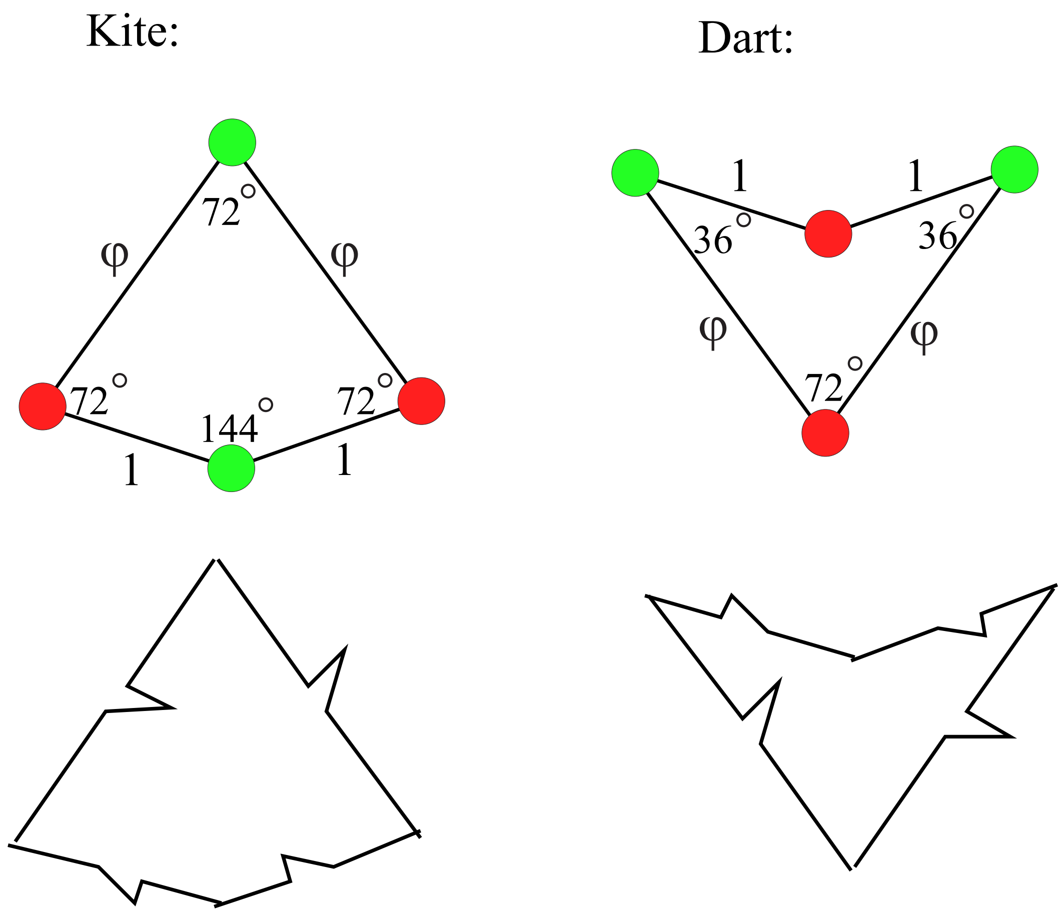
The Penrose tiles are obtained by cutting in two a rhombus that has a 72° angle. The resulting quadrilaterals have edges in the ratio

$$\varphi = (1 + \sqrt{5})/2 = 1.618\ldots,$$

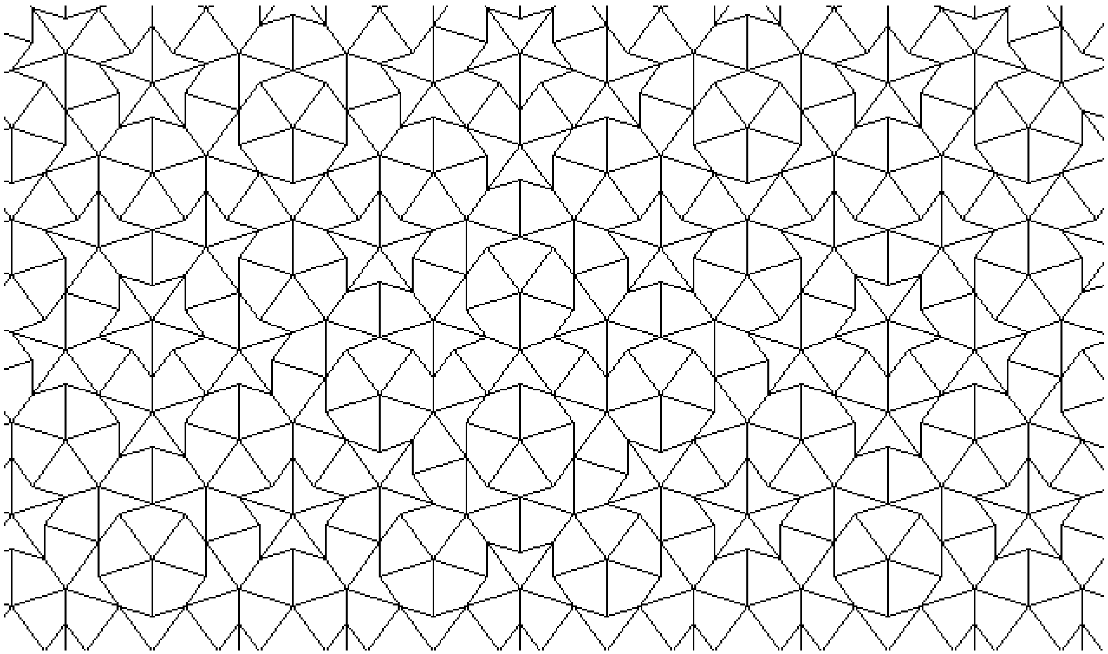
the golden ratio.

The vertices are colored red and green. Valid tilings are **edge-to-edge** and the **colors at the vertices** must match. (This condition prevents one from gluing the kite and dart back together to form the rhombus.)

These matching rules can be easily enforced using geometric constraints using bumps and dents:



Here is a part of a tiling using kites and darts. (For clarity, the bumps and the dents are not shown):



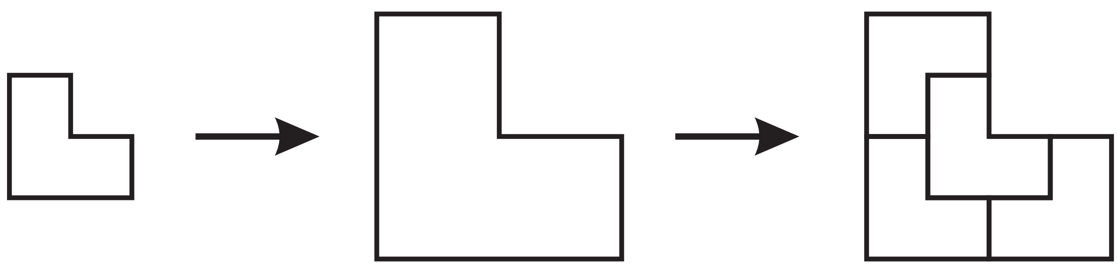
The following result is proved in the homeworks:

Theorem. Penrose kite and dart are an aperiodic pair of prototiles. They admit valid tilings with a 5-fold rotational symmetry.

Substitutions

One can generate valid tilings by Penrose kites and darts using **substitutions**, as seen at the exercises.

With substitutions one can easily generate hierarchical, non-periodic tilings. We illustrate this with a simple example, the **chair substitution**, where an ***L*-tromino** is expanded by factor two and cut into four smaller L-trominoes:



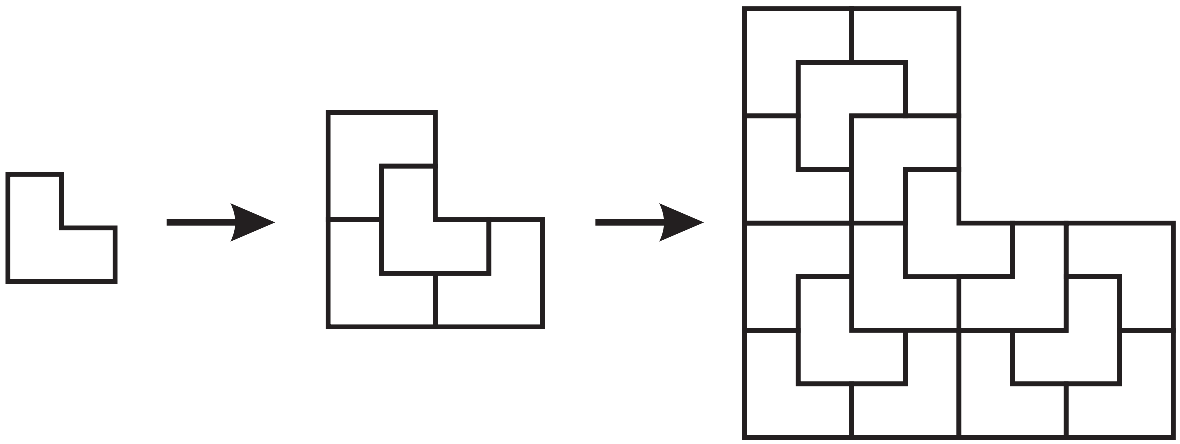
Let us call this patch of four tiles a **macro tile**.

So, starting from a single tile, we repeat the following operations:

(i) **Magnify** your patch of tiles by factor two horizontally and vertically.

(ii) **Substitute** each magnified tile by four smaller copies as above.

Here are the first two iterations:

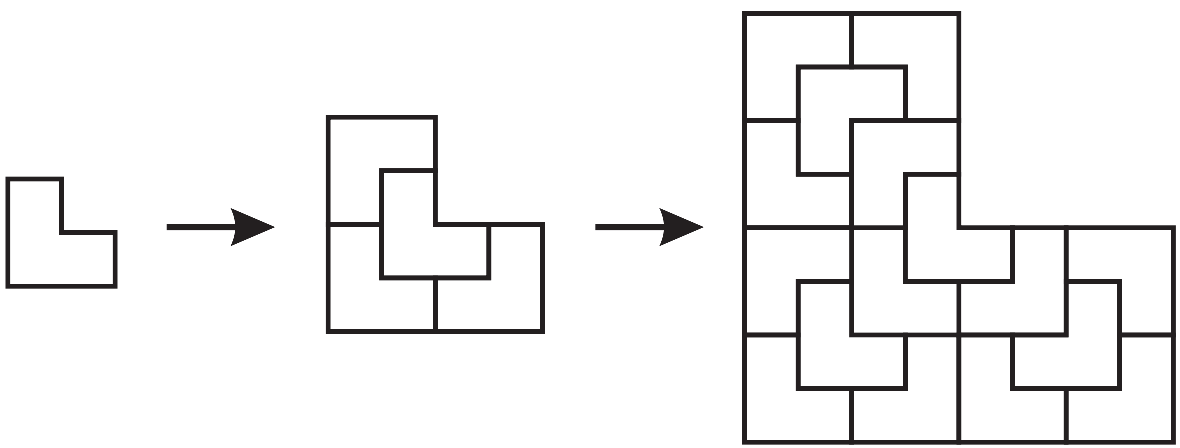


The k'th iteration results in a patch of 4^k tiles.

Iterating the substitution provide ever larger areas covered by the tiles. To obtain a tiling of the infinite plane, we suitably **position** the obtained patches on the plane so that the next patch **extends** around the previous one, and take the **limit** of the process.

We do the positioning in such a way that the patches grow in all directions, so that each point of the plane gets eventually covered by a tile.

For example, we can align the second iteration over the initial tile so that it is fully surrounded in the patch, and repeat this positioning at all even rounds.



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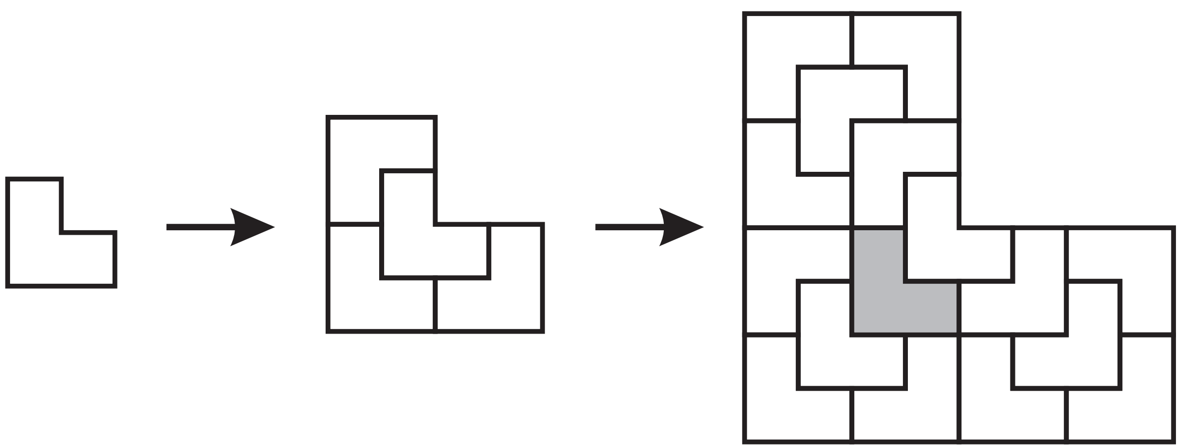
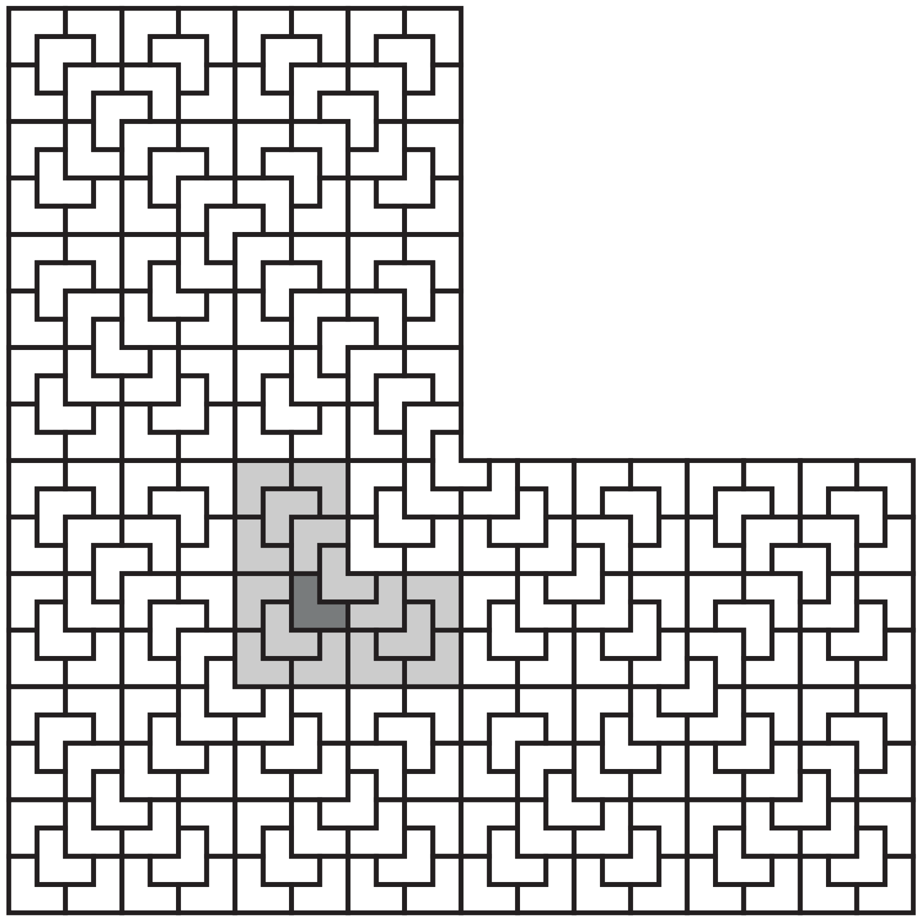


Because the grey tile is not on the boundary, it is guaranteed to get surrounded by more and more tiles on all sides. Here's the fourth iterate:



The dark tile is the seed, the light grey part is the patch after two iterations.

In the limit we obtain a **tiling** t of the infinite plane.

In t each tile belongs to a macro tile, which in turn belongs to a macro tile of macro tiles, etc.

To prove that the tiling t is not periodic, we need to know that every tile belongs to a **unique macro tile**.

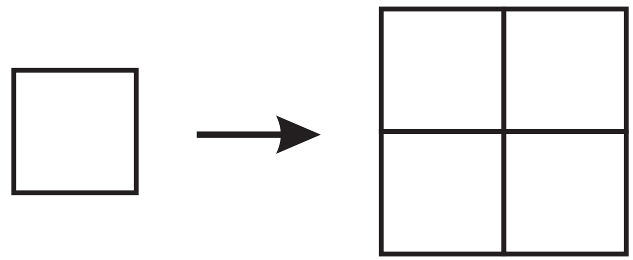
Indeed, two macro tiles cannot overlap:

We conclude that tiling t has a unique partitioning into a tiling by the macro tiles. The same reasoning then applies to the next levels, so that tiling t can be partitioned in a unique way into a tiling by k'th level macro tiles, for every k.

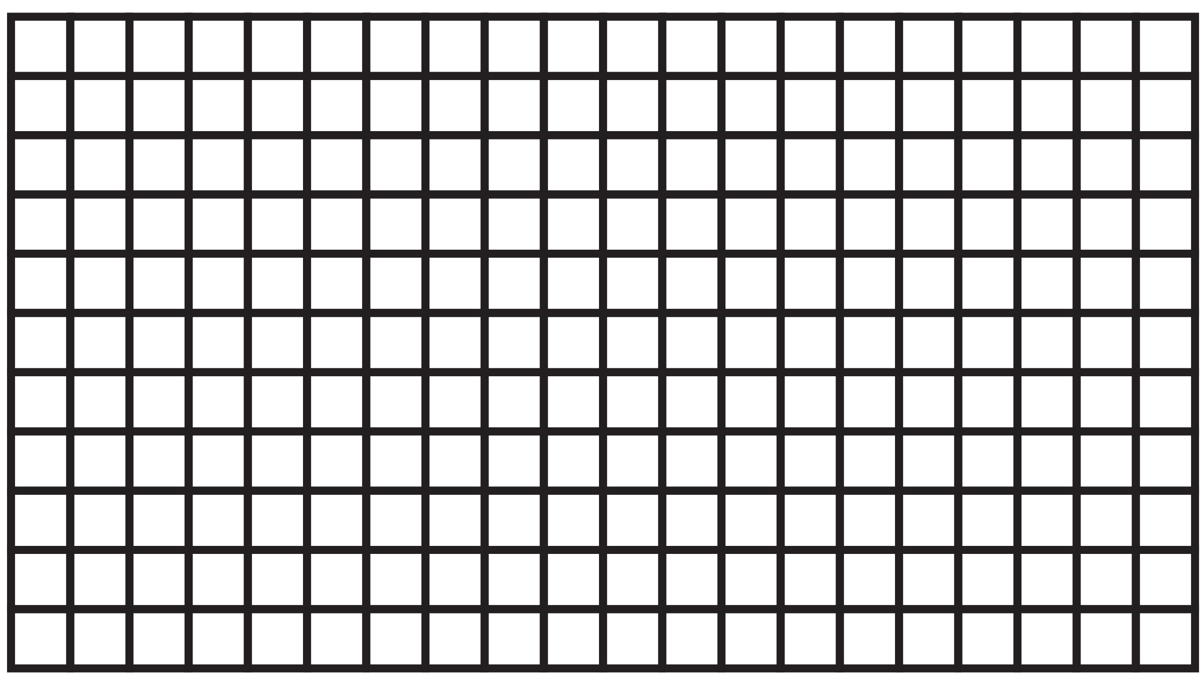
This implies that tiling t is **not periodic**:

We have proved that the tiling obtained by iterating the chair substitution is non-periodic. It was essential in the proof that the partitioning of the tiling into macro tiles is **unique**.

Example. Also the **square substitution**



can be iterated to generate a tiling such that each tile belongs a macro tile, macro-macro tile, etc.:



How come this tiling then is periodic??

Remark: the L-tromino is of course not aperiodic.

There are general methods to decorate tiles with color constraints in substitutions so that non-periodicity is forced. This increases the total number of tiles.