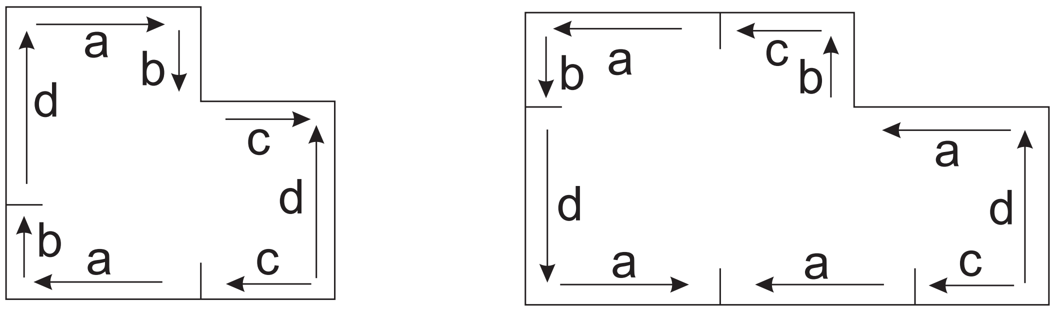
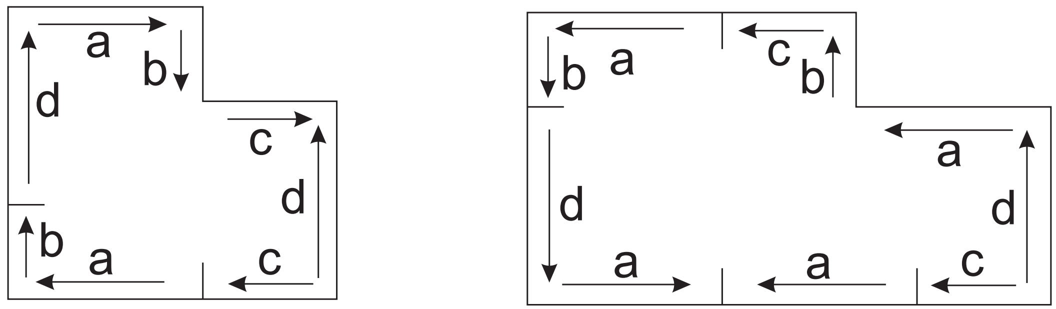
Amman's aperiodic tile set

The following pair of tiles, due to R.Amman in 1977, forms an aperiodic tile set.



The tiles may be rotated and flipped in any orientation. Let us call these the **A-tile** and the **B-tile**.

The **labeled arrows** along the edges give a matching rule: each arrow must fit against an arrow with the same label and orientation.

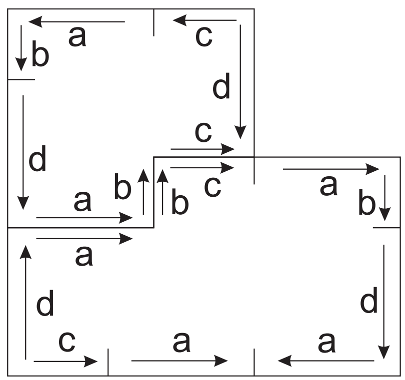


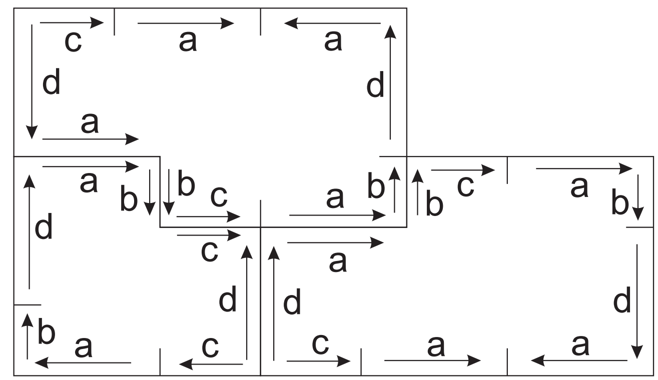
The lengths of the arrows are arbitrary (positive), but all arrows with the same label have also the same length.

Remark: this matching rule can not be implemented geometrically with bumps and dents. The reason is that some tiles are flipped, in which case the bumps of the matching arrows would line up against each other.

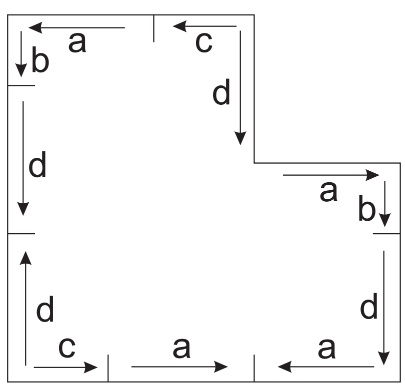
The tiles can be implemented geometrically, though, by not using bumps but only dents and a new key tile that fits between lined up dents. This way the tiles can be converted into a aperiodic set of **three geometric tiles**.

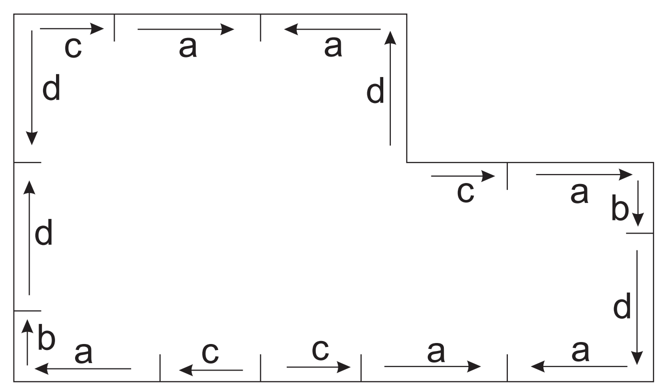
An A-tile and a B-tile fit together into the macro tile on the left, and an A-tile and two B-tiles form also the macro tile on the right:





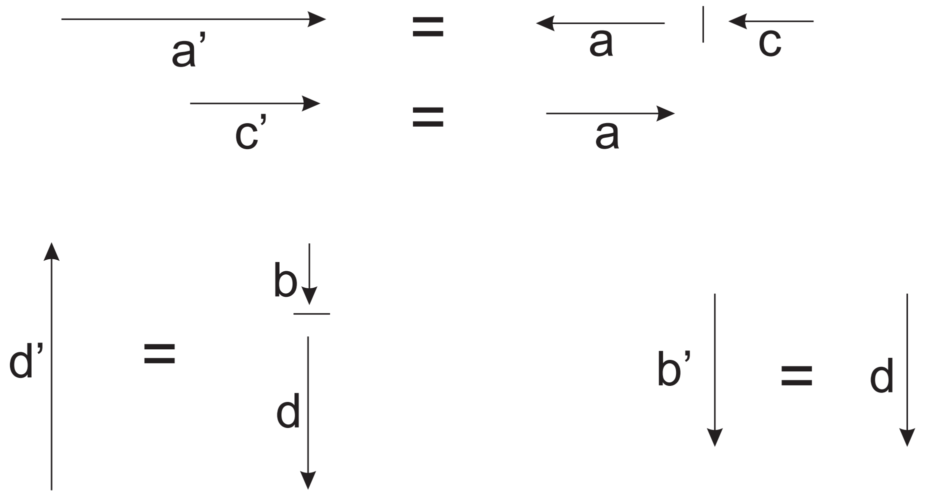
Any tiling by these **super-A** and **super-B** tiles



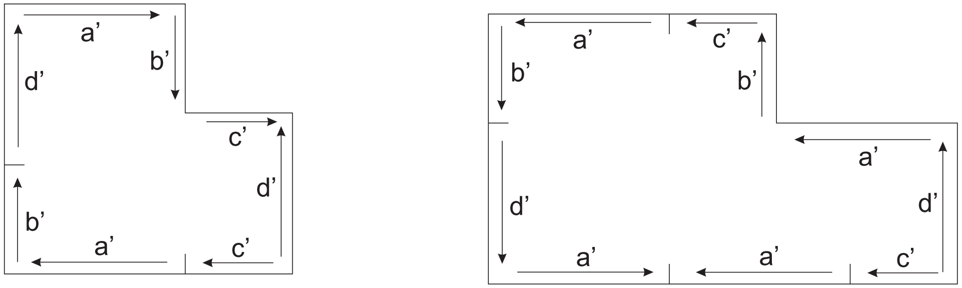


can be broken into a tiling by the original tiles.

Let us redecorate the supertiles with arrows labeled a', b', c' and d', where the new arrows represent combinations of old arrows as follows:



The redecorated supertiles



are called **expanded** A and **expanded** B, respectively.

- In any tiling by the expanded tiles, the tiles can be replaced by the corresponding supertiles, and the tiling remains valid.
- Conversely, in any tiling by the supertiles, replacing the supertiles by the corresponding expanded tiles yields a valid tiling.

(The latter fact follows from the fact that in the supertiles each c-arrow is always immediately followed by an a-arrow in the same direction, and every b-arrow is immediately followed by a d-arrow.)

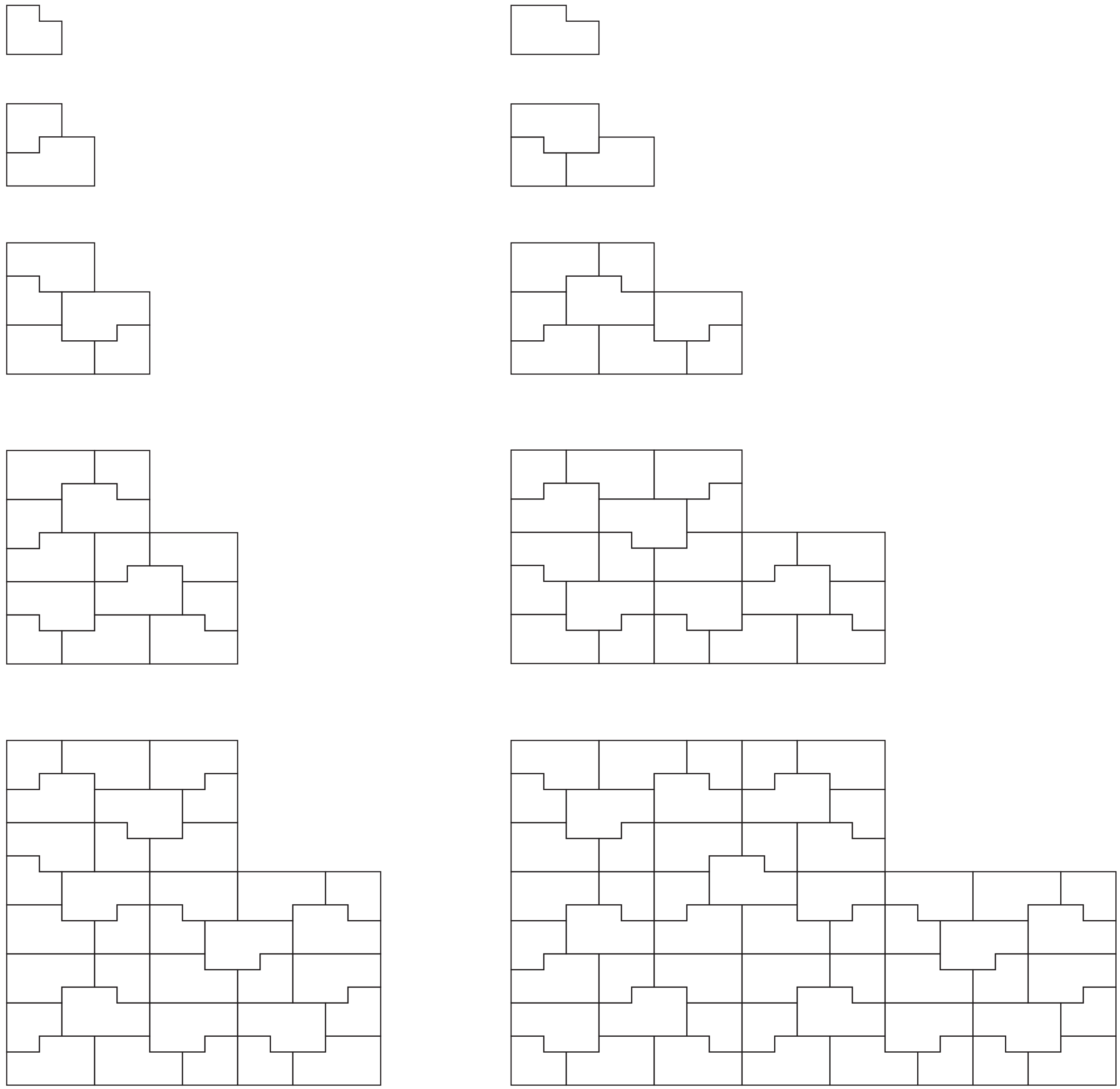
We conclude that the **supertiles** and the **expanded tiles** admit exactly the **equivalent tilings**.

The expanded tiles are **isomorphic** to the original tiles, where the arrows with labels a', b', c' and d' correspond to the arrows a, b, c and d, respectively. (However, the arrow lengths and their ratios may change, so the shapes of the expanded tiles are not necessarily similar to the original tiles.)

We can now build supertiles of **level two** by combining the expanded tiles the same way we combined the original tiles to build the first level supertiles.

Iterating the process allows us to build supertiles and expanded tiles of levels two, three, four and so on. These provide tilings of larger and larger regions of the plane by the original A- and B-tiles.

We can take a **limit**, which yields a valid, hierarchical **tiling** of the infinite plane.



Next we prove that there are **no periodic tilings**.

Consider an arbitrary tiling t of the plane by A and B.

Claim 1: Every tile belongs to a supertile.

Claim 2: The A- and B-tiles can be grouped into non-overlapping supertiles.

Claim 3: The grouping into supertiles is unique.

Theorem. The A- and B-tiles form an aperiodic pair of tiles.

Proof.

The extension theorem

Taking a limit, in all our substitutions we were able to position the patches so that level k patch is a sub-patch of the level $k+1$ patch that extends to all directions. Then it is clear that there is a “limit” tiling that has all the levels as sub-patches.

But it can be shown that even if the obtained patches do not contain previous ones as sub-patches, a valid tiling exists as long as arbitrarily large disks can be covered. This extends the **compactness argument** of Wang tiles to geometric tiles.

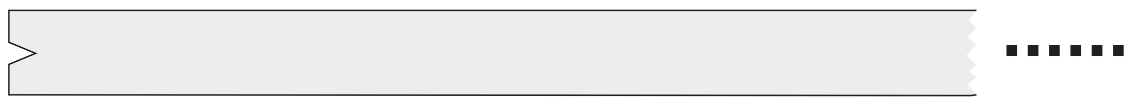
Theorem (extension theorem). A finite protoset $\mathcal{P}$ of topological disks admits a tiling if and only if, for every $r > 0$, a disk of radius r can be covered by copies of the prototiles. More precisely: there is a collection of tiles, all congruent to elements of $\mathcal{P}$, such that

(i) the interiors of the tiles are pairwise disjoint, and

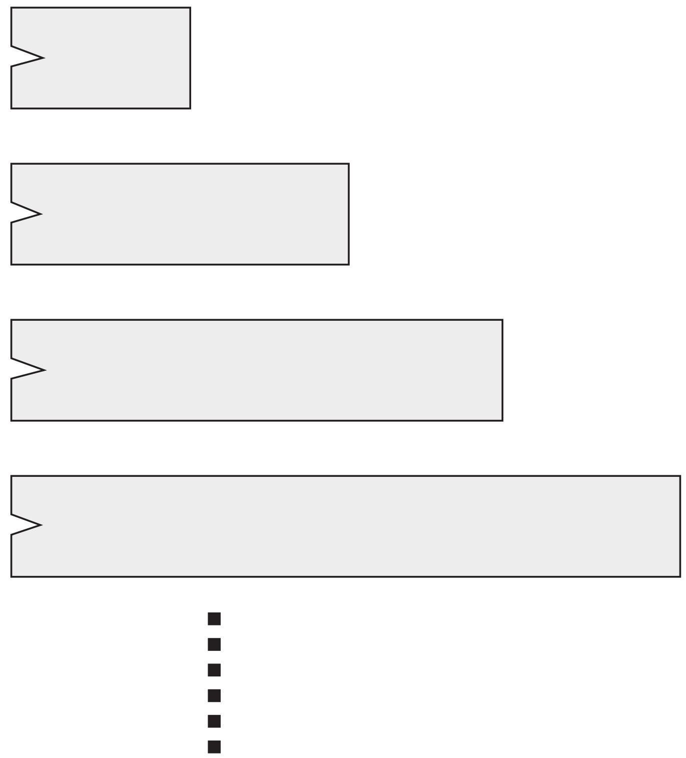
(ii) a disk of radius r is included in the union of the tiles.

Remark: The theorem only considers **finite** protosets of **topological disks**.

Example. The following single "tile" tiles arbitrarily large disks but does not tile the plane. (The tile is not a topological disk.)



Example. The following infinite prototile set (all topological disks) tiles arbitrarily large disks but does not tile the plane.



The periodicity theorem

For Wang tiles we have proved that **"One-periodic $\Longrightarrow$ two-periodic"**:

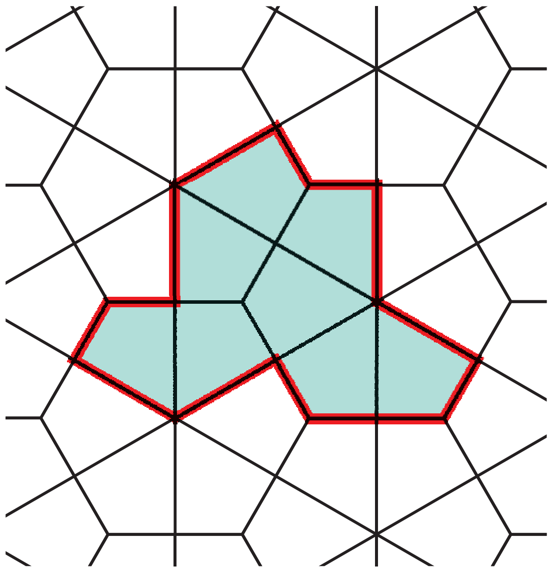
Theorem. If a Wang tile set admits a tiling with a period, then it also admits a tiling with two periods in non-parallel directions.

Also this can be generalized to geometric tiles. What we require is that:

- Tiles are polygons,
- Considered tilings are edge-to-edge,
- The prototile set is finite.

Theorem (periodicity theorem). Let $\mathcal{P}$ be a finite set of polygons. Assume that there exists an edge-to-edge tiling by the protoset $\mathcal{P}$ that is one-way periodic (=invariant under some translation). Then there also exists an edge-to-edge tiling by $\mathcal{P}$ that is two-way periodic (=invariant under translations by two linearly independent vectors).

Hat: an aperiodic monotile

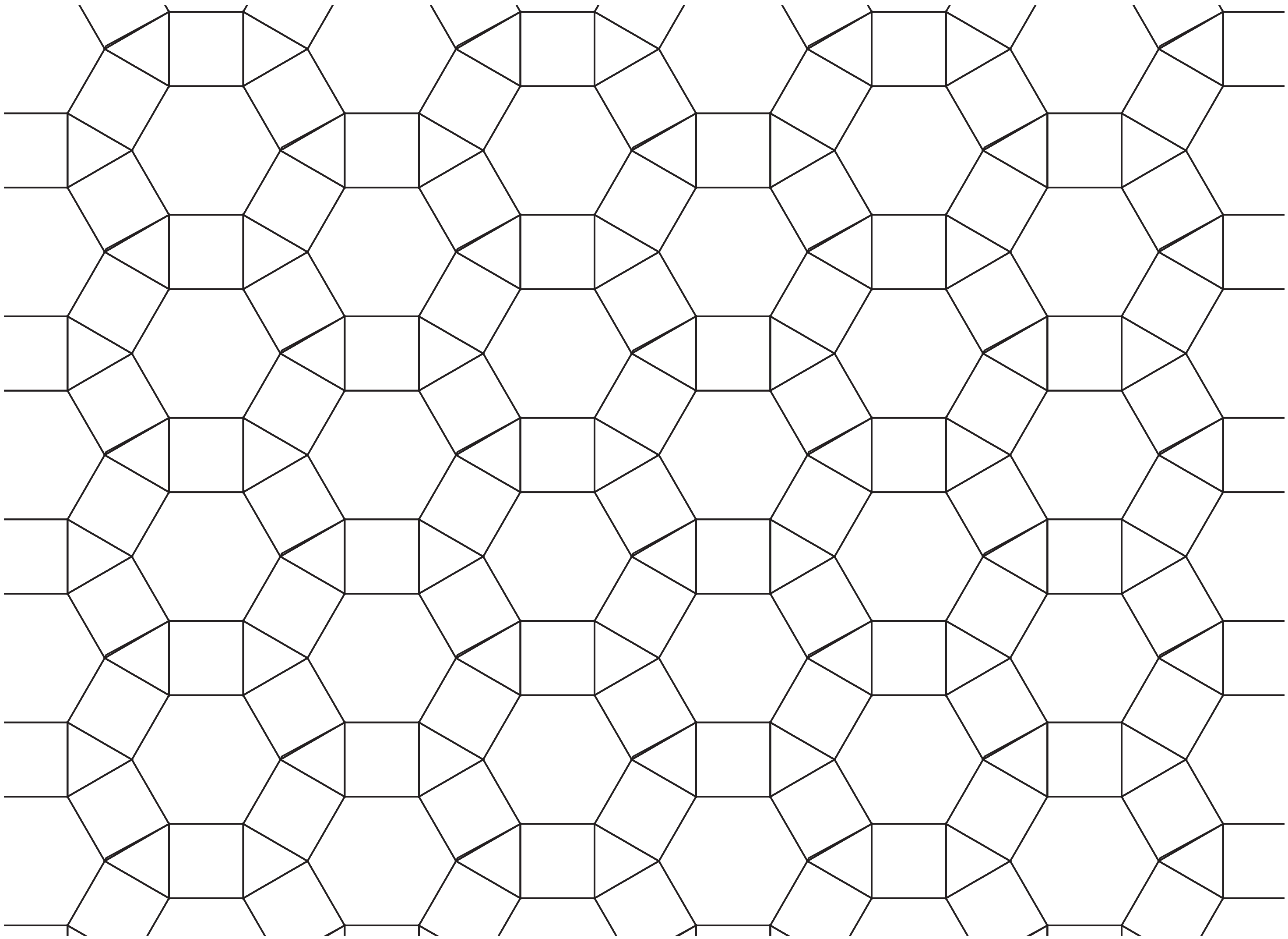


The hat is an **aperiodic monotile**: there exists monohedral tilings of the plane where all tiles are congruent to the hat, but none of these tilings are periodic (=invariant under a non-zero translation).

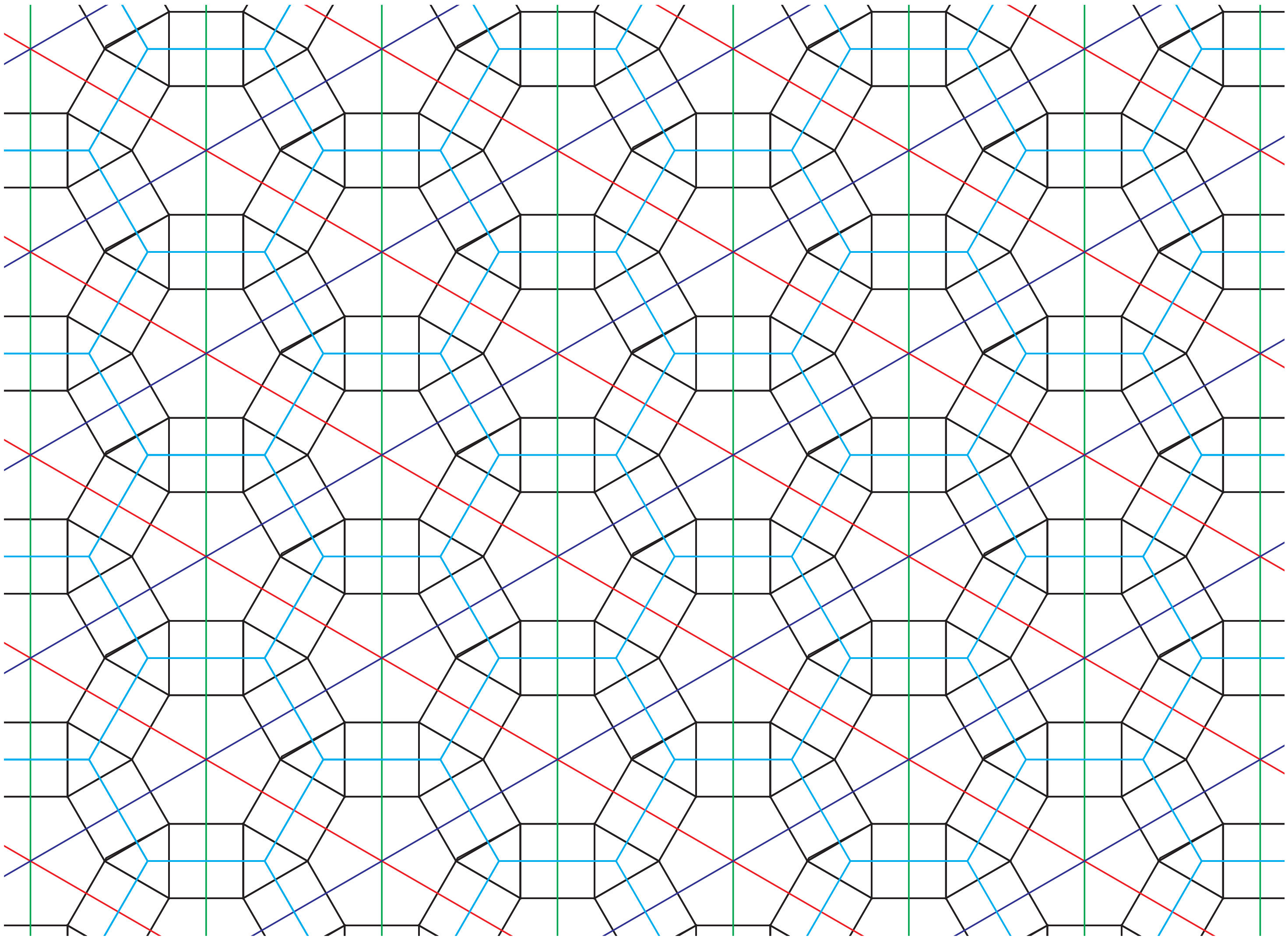
In the tilings both even and odd isometric copes of the hat are used (=the hat may be flipped upside down).

The hat “lives” in the grid that is formed by overlapping the regular tilings by equilateral triangles and regular hexagons.

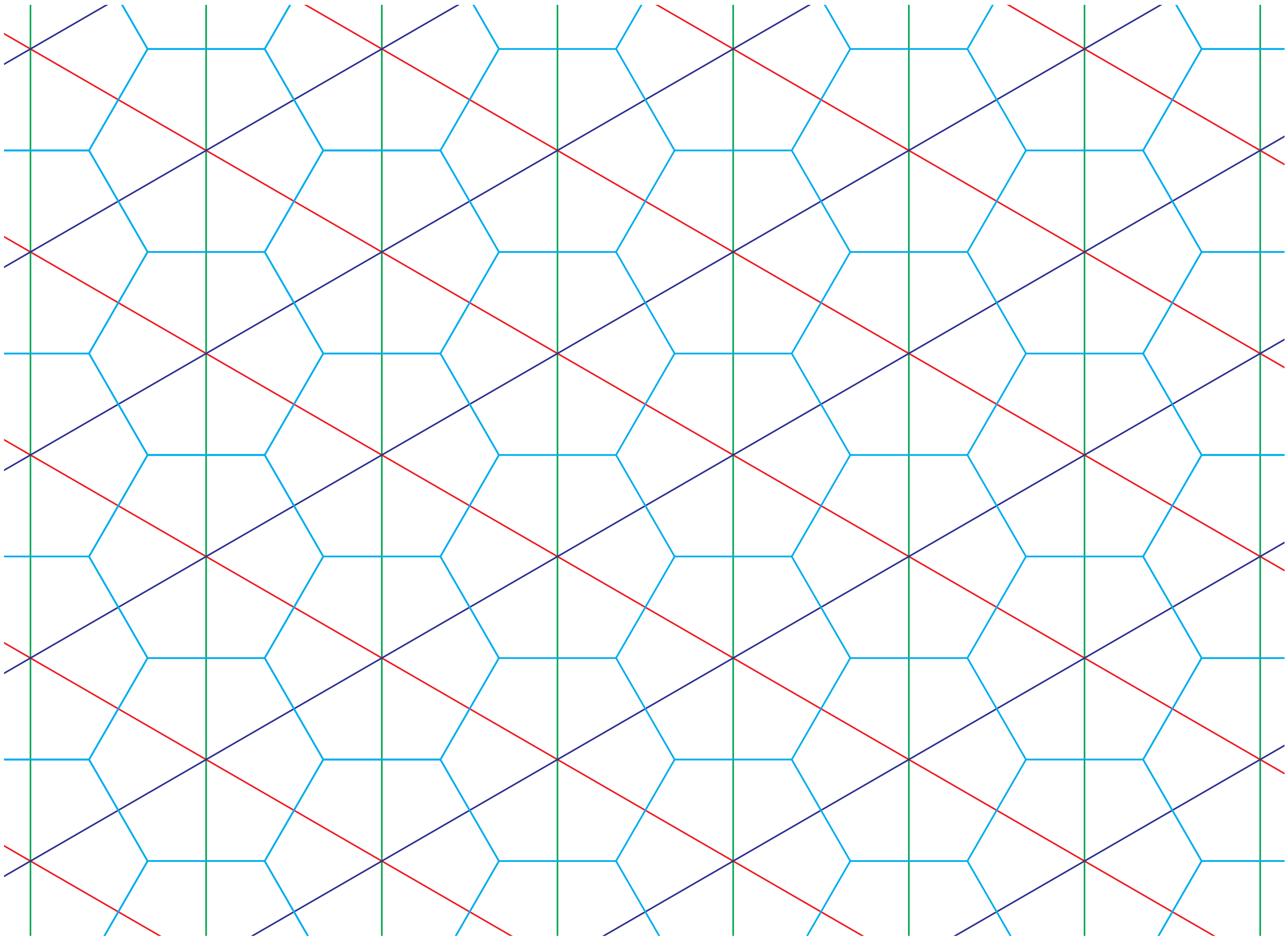
The grid is the dual of the $3 \cdot 4 \cdot 6 \cdot 4$ Archimedean tiling, obtained by joining the centers of adjacent tiles of the Archimedean tiling.

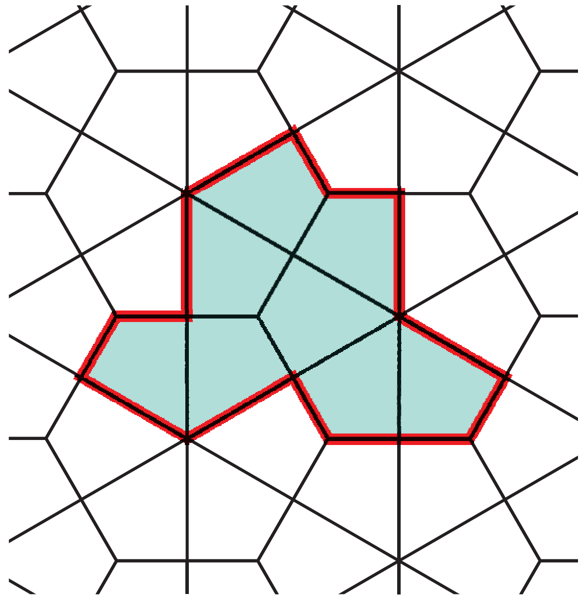


The grid is the dual of the $3 \cdot 4 \cdot 6 \cdot 4$ Archimedean tiling, obtained by joining the centers of adjacent tiles of the Archimedean tiling.



The grid is the dual of the $3 \cdot 4 \cdot 6 \cdot 4$ Archimedean tiling, obtained by joining the centers of adjacent tiles of the Archimedean tiling.



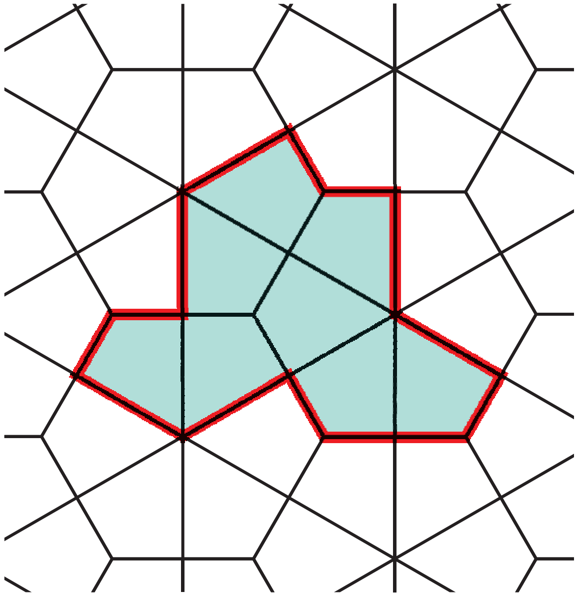


A kite in the grid has

- two short edges (length 1),
- two long edges (length $\sqrt{3}$).

The hat inherits these edge lengths. As a polygon the hat has also one edge of length 2 (formed by two consecutive parallel short edges of kites), but this segment is considered as two edges of length 1.

$\Longrightarrow$ The hat has 6 long edges and 8 short edges.



Reading the edges as vectors, going around the hat clockwise, gives 6 long vectors and 8 short vectors that sum up to $\vec{0}$.

But: also the sum of the long vectors is $\vec{0}$ and the sum f the short vectors is $\vec{0}$.

Reason: The six long vectors are three pairs of opposite vectors.