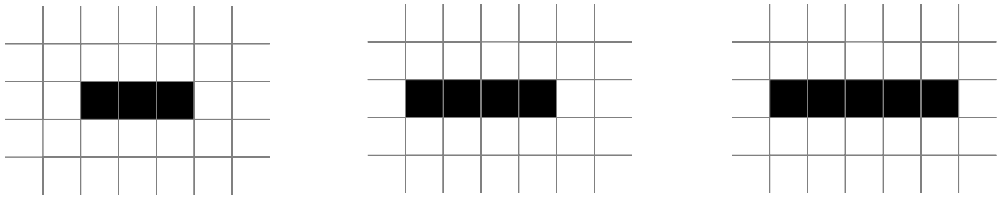
Cellular Automata. Homework 1 (26.1.2026)

1. Calculate the orbits of the following three patterns in *Game-of-life*. Determine if they are fixed points, periodic, eventually fixed, or eventually periodic. You may use any online Game-of-life simulator.



(a) (b) (c)

   (a) Three living cells in line.
   (b) Four living cells in line.
   (c) Five living cells in line.

2. Let G be the *xor* CA of Example 1 in the notes.

   (a) Prove that every configuration c has exactly two pre-images, i.e. two configurations a and b such that $G(a) = c$ and $G(b) = c$.
   (b) Prove that if a and b are two different finite configurations (when 0 is the quiescent state) then $G(a) \neq G(b)$.

3. Let G be the *xor* CA of Example 1 in the notes. Let c be the configuration $c(0) = 1$ and $c(k) = 0$ for all $k \neq 0$. Prove that the state of cell k in $G^t(c)$ is 1 if and only if

   (i) $-t \leq k \leq 0$, and
   (ii) the binomial coefficient $\begin{pmatrix} t \\ -k \end{pmatrix}$ is an odd number.

4. Let G be the *xor* CA of Example 1 in the notes.

   (a) Determine all fixed points of G and all fixed points of G^3.
   (b) Show that for every $n \in \mathbb{Z}_+$, CA G can have at most 2^n periodic points with period n.

5. (a) Prove Proposition 1: The composition of two CA functions is a CA function.
   (b) Determine the Wolfram number of the composition of elementary CA 102 and 60. (These are the *xor* CA and its mirror image.) What is the minimal neighborhood of the composition ?

6. Let G be the elementary CA whose local rule is the majority rule: $f(a, b, c) = 1$ if and only if $a + b + c \geq 2$.

   (a) What is the Wolfram number of this CA ?
   (b) Prove that if the state of a cell remains unchanged in two consecutive generations then it remains unchanged from that moment on.
   (c) Prove that under iterations of G^2 the state of each cell changes at most once.

7. Prove that every strongly periodic configuration is eventually periodic.