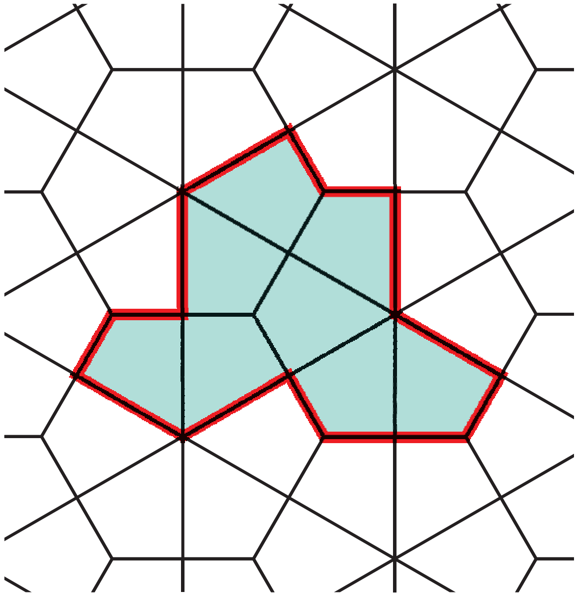


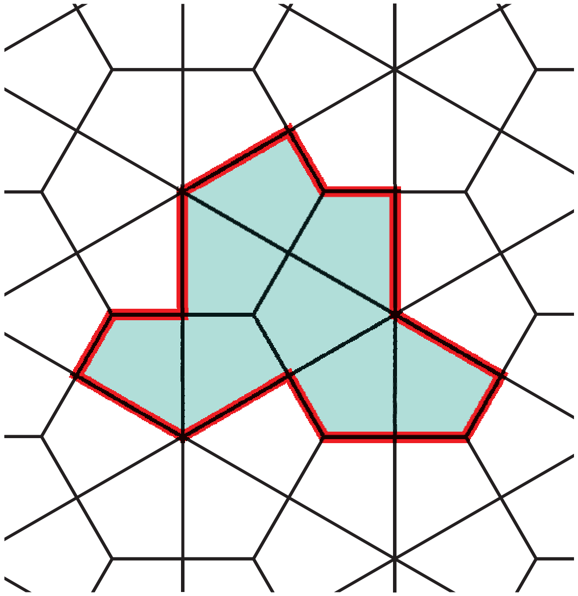
Thus one can **deform** the hat by scaling the lengths of

- all long edges by some $a \geq 0$ and
- all short edges by some $b \geq 0$.

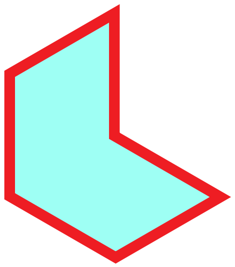
(but not $a = b = 0$)

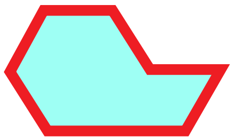
The scaled long edge vectors sum up to zero and and the scaled short edge vectors sum up to zero.

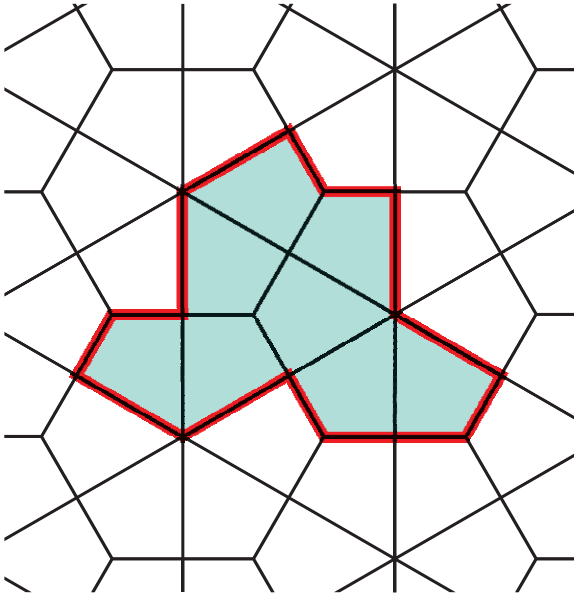
$\Longrightarrow$ The scaled edges define a boundary of a deformed tile.



We especially need the extremal cases where $a = 1, b = 0$; or $a = 0, b = 1$:







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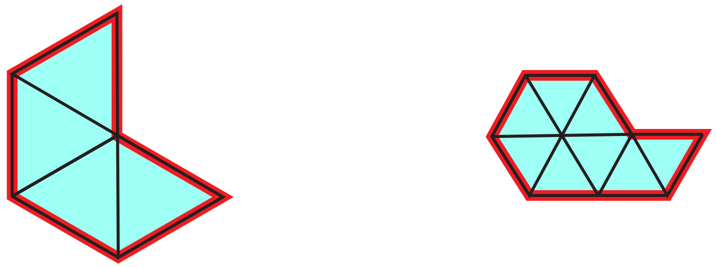


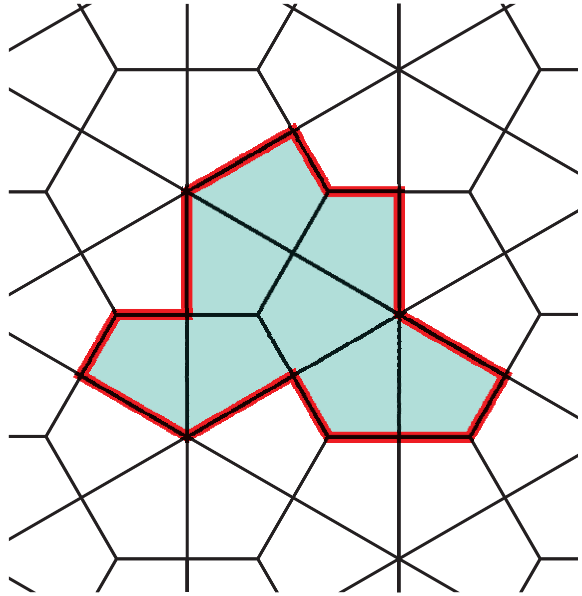
In fact we only use the left one, **chevron**.

Deforming the tiles in a valid hat tiling produces a corresponding **valid tiling by deformed tiles**:

Let $\mathcal{T}$ be a hat tiling, and let $\mathcal{V} \subseteq \mathbb{R}^2$ be its vertex set.
Assume w.l.o.g. that $\vec{0} \in \mathcal{V}$.

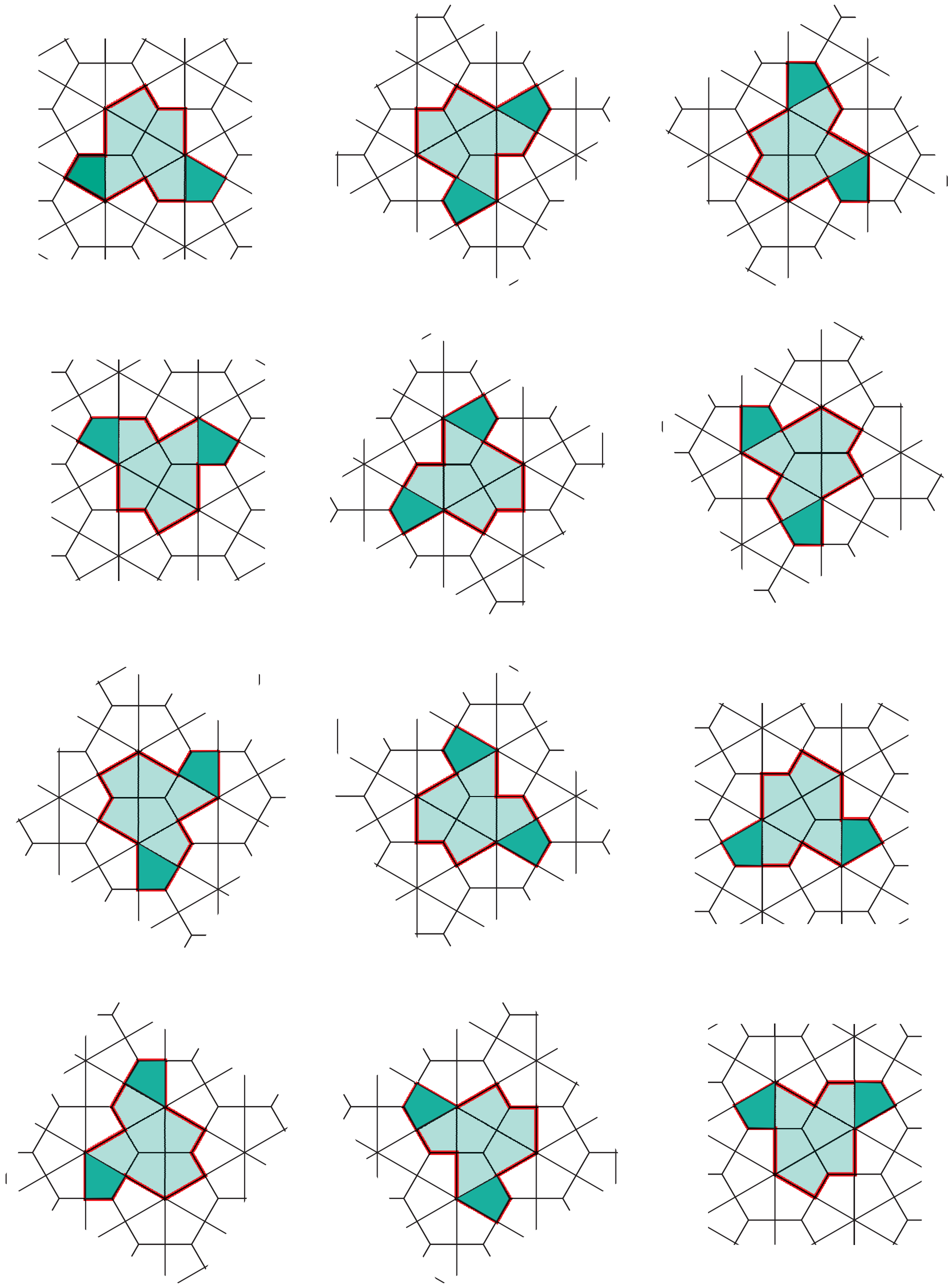
• Along any cycle that follows the edges in the tiling: the sum of the long vectors is $\vec{0}$ and the sum of the short vectors is $\vec{0}$.

• In the deformed tiling, vertex $V \in \mathcal{V}$ will be moved to position $V' \in \mathbb{R}^2$ as follows: Take any path along the edges from $\vec{0}$ to V. Scale the long and short edges along this path by factors a and b. The path still starts at $\vec{0}$ but the new end point will be V'. The position of V' does not depend on the choice of the path!



It turns out (case analysis, details skipped) that hats in any valid tiling are aligned on the underlying grid of kites. (It is enough to prove that surrounding hats of any given hat are aligned in the same grid: by induction then all hats are aligned.)

$\Longrightarrow$ there are **12 available orientations of the hat**: six by rotation, and another six by rotating the flipped over hat.



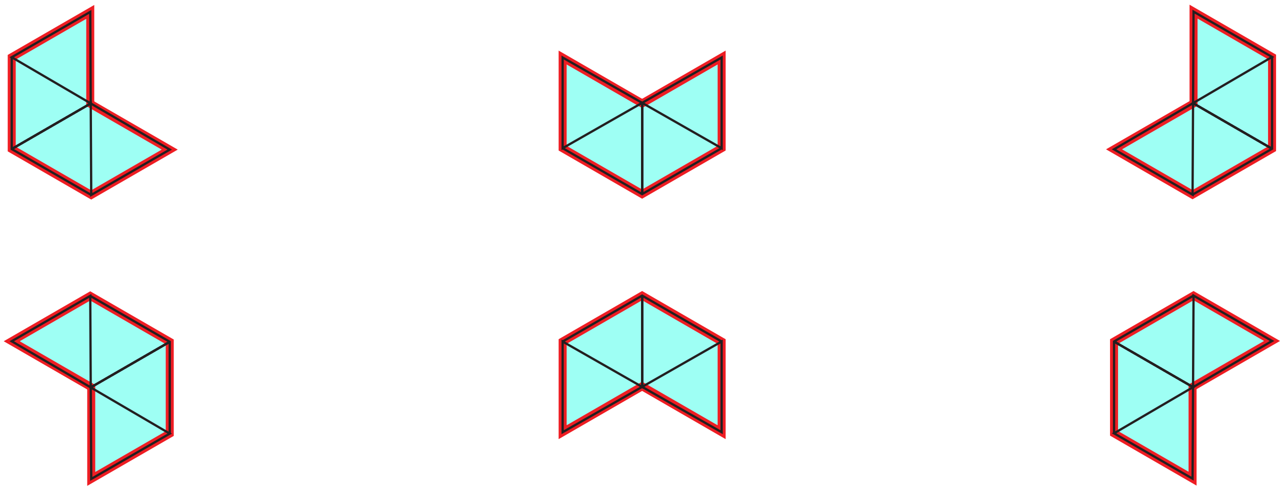
In the hat, one pair of oppositely oriented kites is covered twice – all other kite orientations are covered once.

The four hat orientations in each column cover the same kite orientation twice.

Because all kite orientations appear with equal proportions in the underlying grid, a valid tiling must have one third of its tiles from each of the three columns.

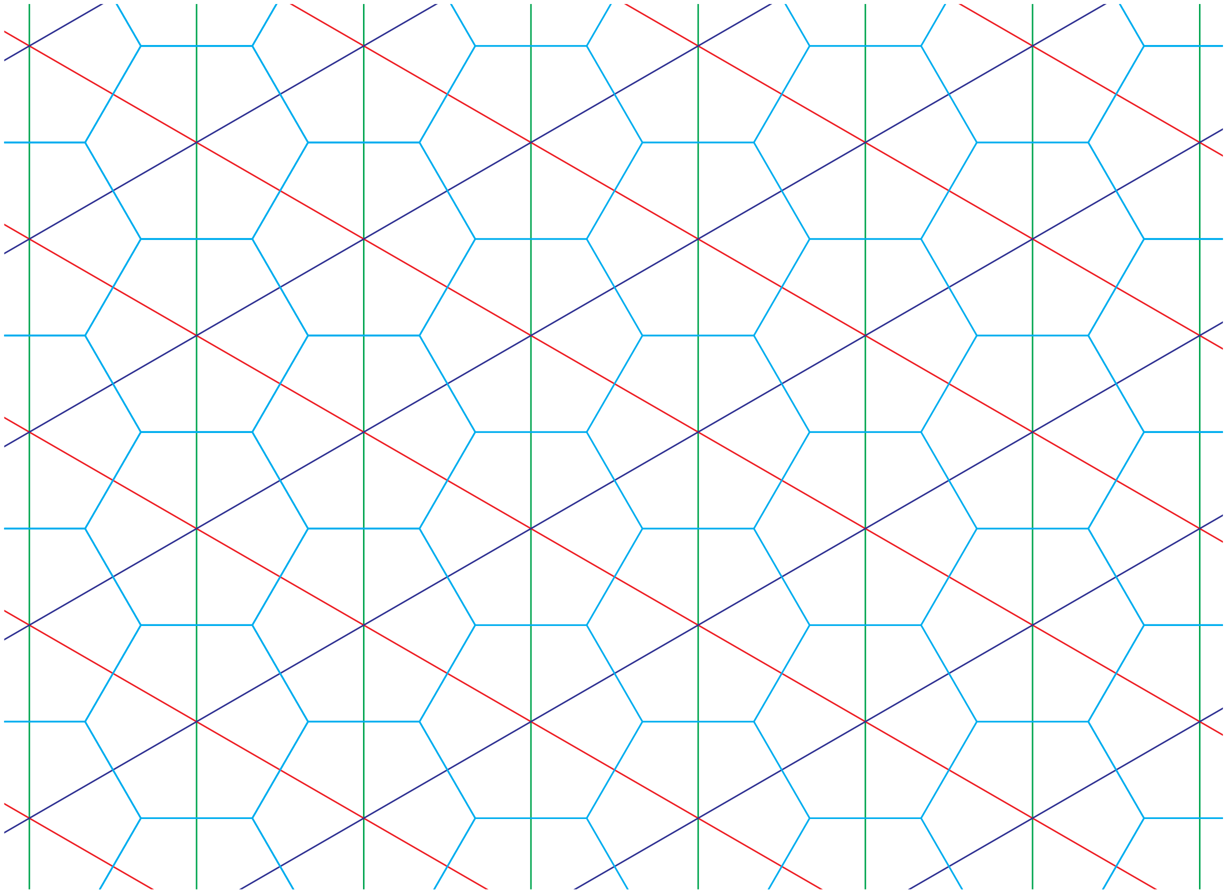
Hat $\longrightarrow$ chevron deformation preserves the orientations of the long edges.

From the 12 orientations of the hat we obtain six different orientation of the chevron:



A chevron tiling that is obtained by deforming a hat tiling has one third of its chevrons from each of the three columns.

The triangular part of the kite grid is formed by three families of parallel lines.



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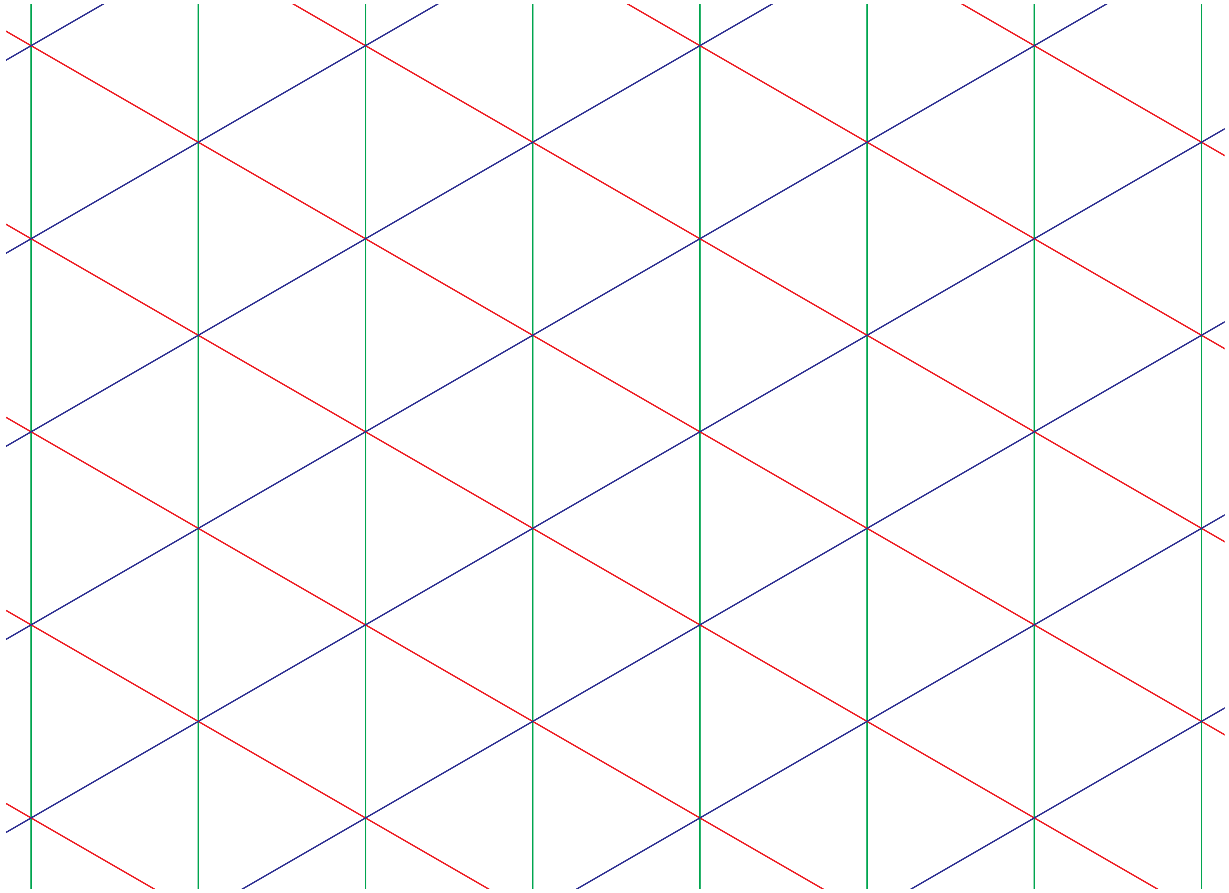
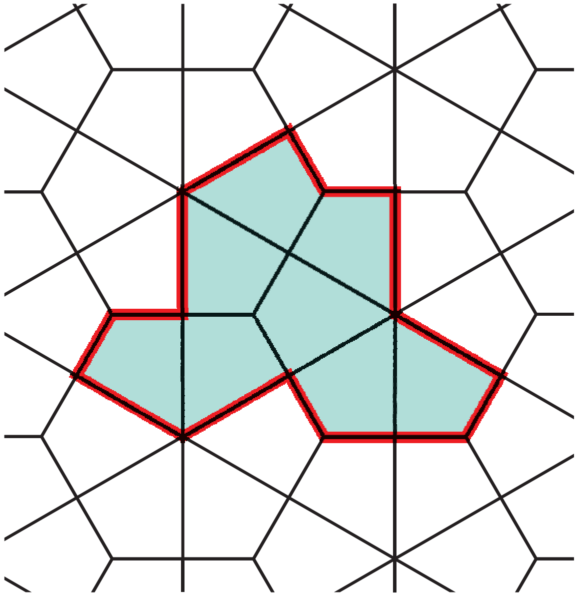


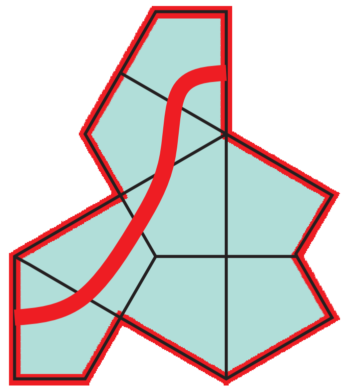
We number the directions 1,2 and 3, and call the lines in direction $i \in \{1, 2, 3\}$ the ***i*-lines** of the grid.

The distance between consecutive i-lines is 3.



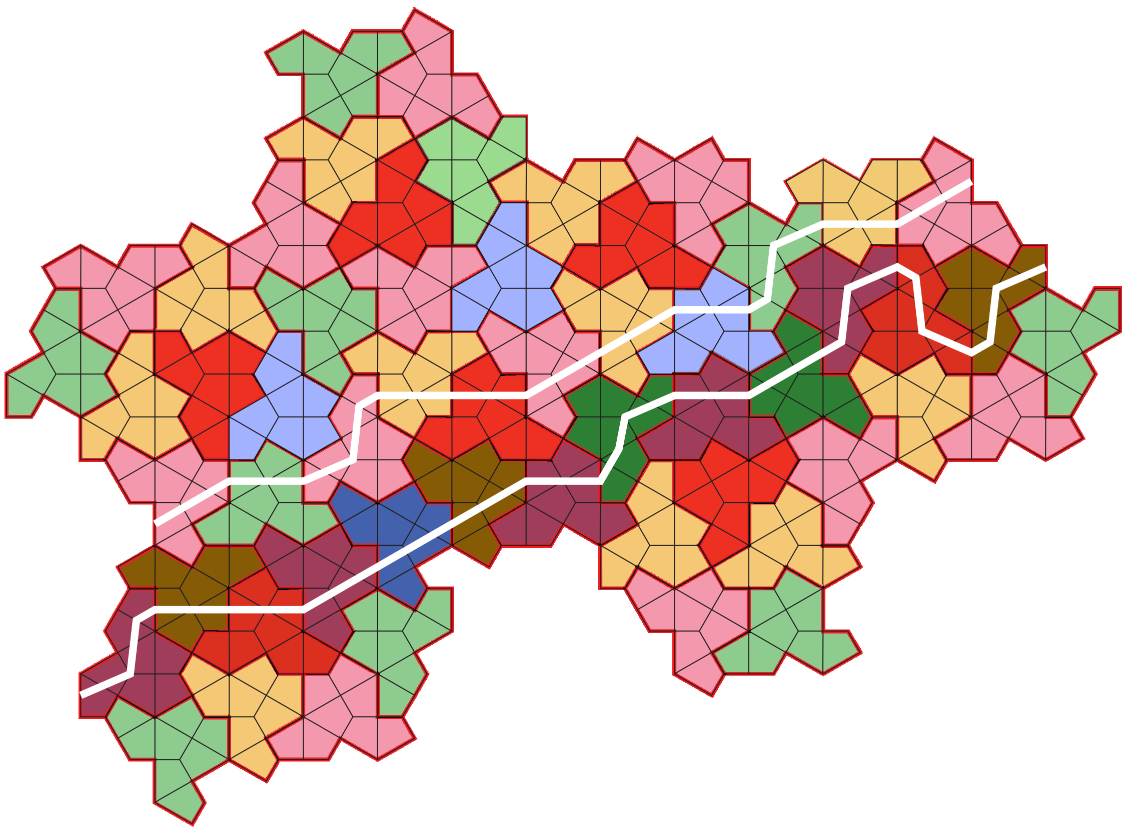
The pairs of parallel long edges of the hat are always on consecutive i-lines.

A **de Bruijn segment** on a hat tile in direction i is a line drawn inside the tile connecting the centers of the two long edges of the hat that are parallel to i-lines.



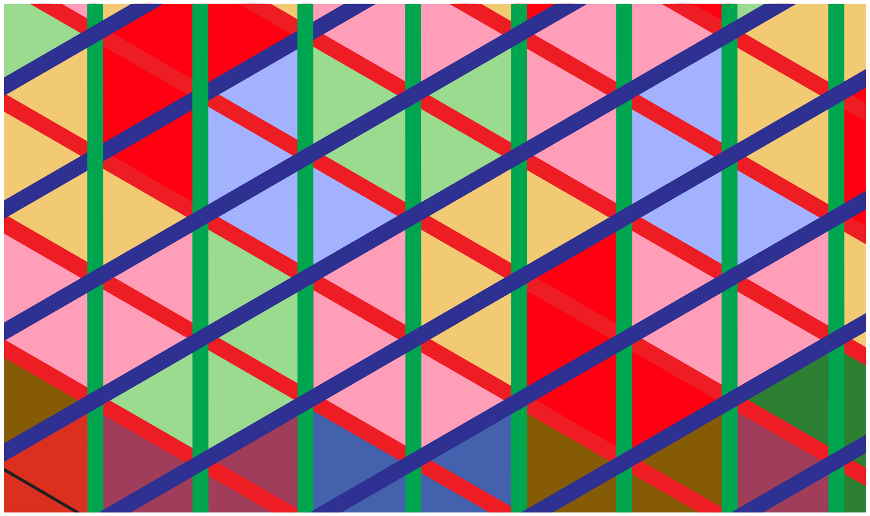
In a hat tiling, de Bruijn segments continue across edges, defining infinite **de Bruijn lines**.

Each tile is crossed by a unique de Bruijn line in each direction i, and the lines in the same direction do not cross each other.

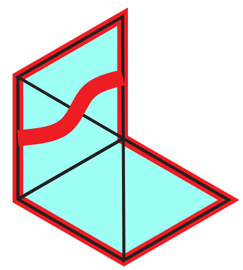


The set of tiles along a de Bruijn line in direction i is called an **i-strip**.

A deformed tiling by chevrons is also aligned on a triangular grid with the same three directions. In this grid the i-lines are at distance $\frac{3}{2}$ from each other, that is, twice as dense as in the kite grid.

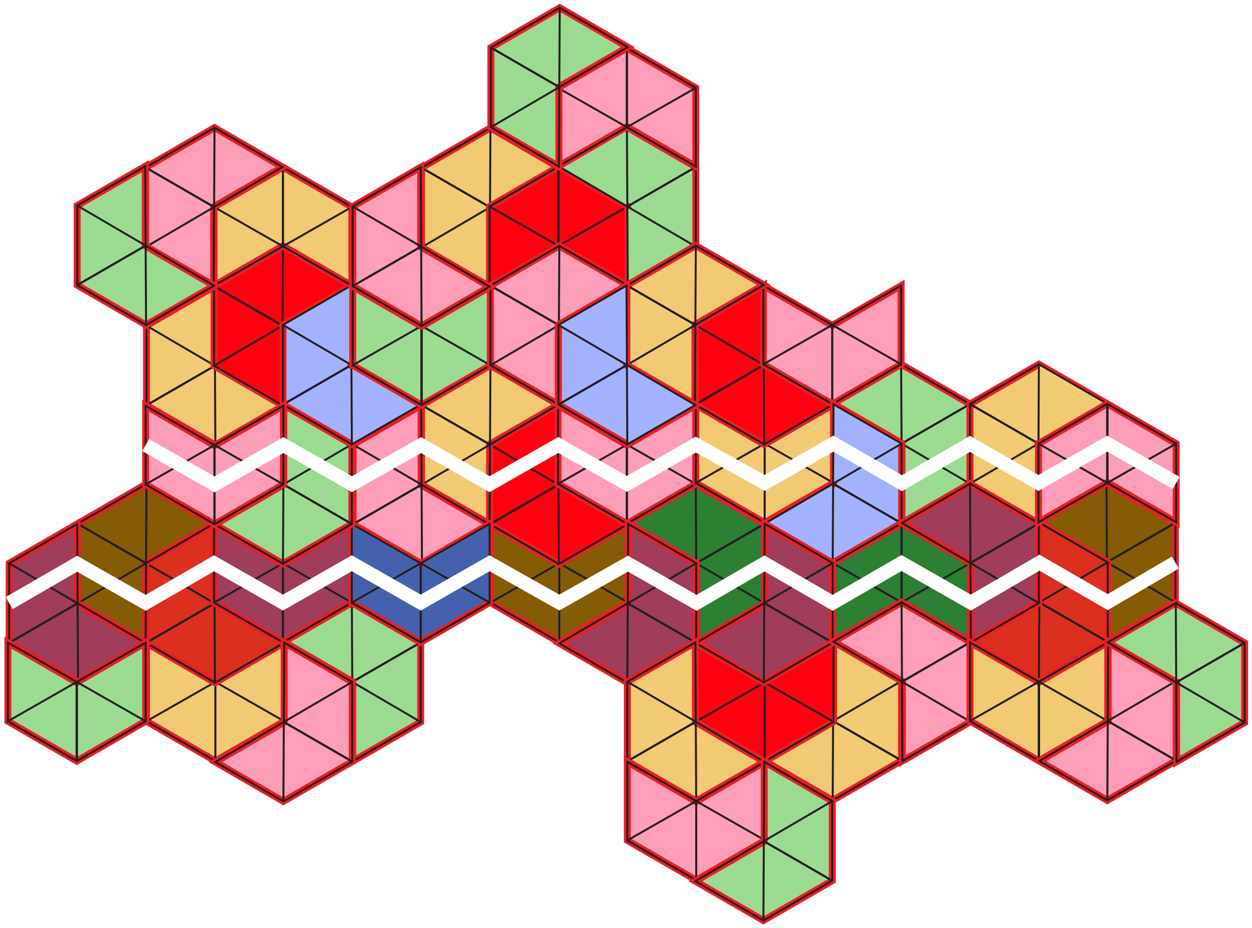


Also the chevron has a pair of parallel edges in each direction. Joining the centers of parallel sides by a line segment gives a **de Bruijn segment**:

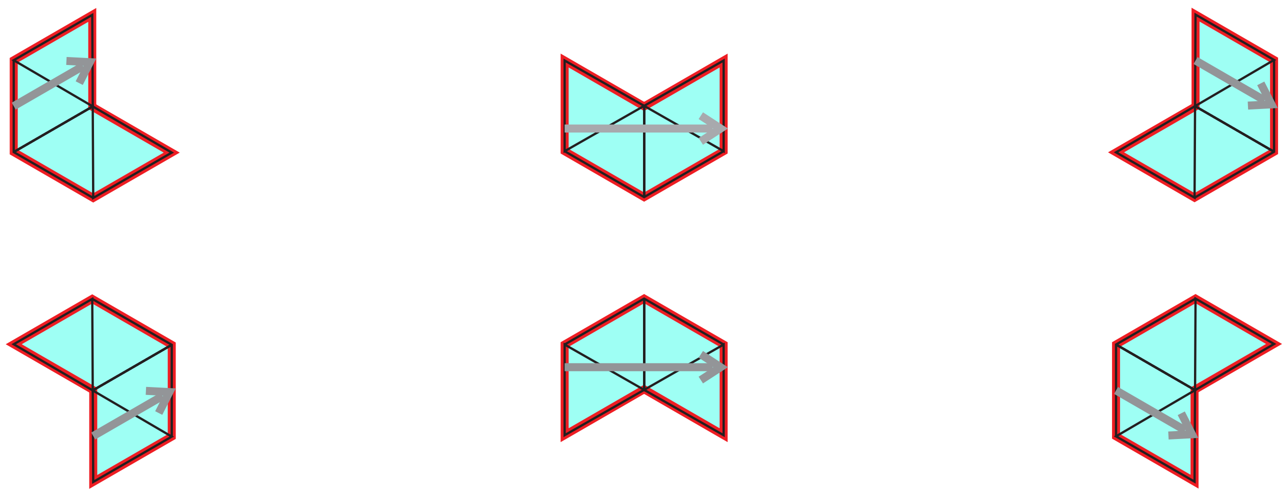


Depending on the orientation of a chevron, the de Bruijn segment either connects consecutive i-lines or it skips over one line.

In a valid tiling the segments define infinite **de Bruijn paths** in directions $i \in \{1, 2, 3\}$, and the tiles along each such path is called an ***i*-strip**.



Recall that (in a tiling by chevrons that is obtained by deforming a hat tiling) one third of the chevrons come from each of the three columns in



Consider, for example, the direction of vertical lines. In one third of the tiles the de Bruijn segment

- moves to the next i-line but goes a step higher (first column),
- moves to the next i-line but goes a step lower (last column),
- skips over one i-line but stays on the same height (middle column).

On the average, the vertical hight remains the same, and the horizontal movement is by $\frac{4}{3}$ lines per tile crossed (=two units since the distance between consecutive lines is $\frac{3}{2}$).

Assume that there exists a two-periodic tiling $\mathcal{T}$ by the hats. Construct the corresponding deformed tiling $\mathcal{T}'$ by chevrons.

Let $\vec{p}$ and $\vec{q}$ be generators of the periods of $\mathcal{T}$, meaning that

$$\mathcal{P} = \mathbb{Z}\vec{p} + \mathbb{Z}\vec{q}$$

is set of the periods of $\mathcal{T}$.

Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}$, and assume that $\vec{0} \in \mathcal{V}$.

Then $\mathcal{P} \subseteq \mathcal{V}$.

Let $\mathcal{V}' \subseteq \mathbb{R}^2$ be the set of vertices of $\mathcal{T}'$, and let

$$f : \mathcal{V} \longrightarrow \mathcal{V}'$$

assign to each vertex $\vec{v}$ of $\mathcal{T}$ the corresponding vertex $f(\vec{v})$ of $\mathcal{T}'$.
(Corresponding means: For any path along edges of $\mathcal{T}$ from $\vec{0}$ to $\vec{v}$ the path obtained by erasing all short edges leads from $\vec{0}$ to $f(\vec{v}$.)

Claim: For any $\vec{v} \in \mathcal{P}$ and $\vec{x} \in \mathcal{V}$ we have that

$$f(\vec{x} + \vec{v}) = f(\vec{x}) + f(\vec{v}).$$

Proof.

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In particular:

- $f(\vec{v})$ is a period of $\mathcal{T}'$ for any $\vec{v} \in \mathcal{P}$, and
- f is a linear on $\mathcal{P}$: $f(i\vec{p} + j\vec{q}) = if(\vec{p}) + jf(\vec{q})$ for all $i, j \in \mathbb{Z}$

Denote

$$\mathcal{P}' = f(\mathcal{P}) = \mathbb{Z}f(\vec{p}) + \mathbb{Z}f(\vec{q}).$$

Elements of $\mathcal{P}'$ are periods of $\mathcal{T}'$ (but there might exist also other periods).

Let $\hat{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the unique linear function that maps

$$\vec{p} \mapsto f(\vec{p}) \text{ and } \vec{q} \mapsto f(\vec{q}).$$

Then $\hat{f}$ and f are identical on $\mathcal{P}$ (but may differ on $\mathcal{V} \setminus \mathcal{P}$).

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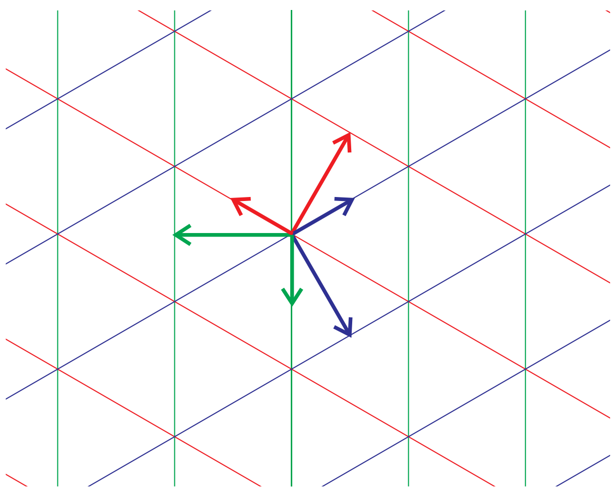
• For each i, let $\vec{v}_i$ be a vector of length 3 perpendicular to i-lines, and let $\vec{u}_i$ be a unit vector parallel to i-lines. We can choose these so that

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_2 = \vec{0} \text{ and } \vec{u}_1 + \vec{u}_2 + \vec{u}_3 = \vec{0}.$$



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• Let τ_i be by vector $\vec{p}_i = k_i \vec{v}_i + y_i \vec{u}_i$. Here $k_i \in \mathbb{Z}$ is the number of tiles that τ_i moves forward on an i-strip, and $y_i \in \mathbb{R}$.

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• We can replace $\vec{p}_i$ by its multiples so we may assume that $k_1 = k_2 = k_3 = k$:

$$\vec{p}_i = k\vec{v}_i + y_i\vec{u}_i$$

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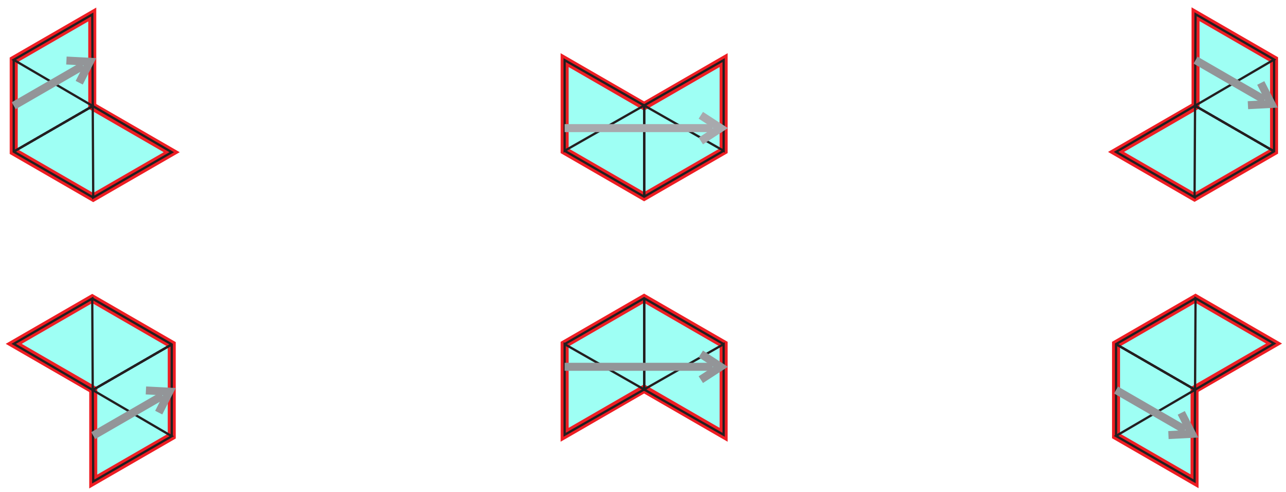

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• By linearity of $\hat{f}$ then

$$\hat{f}(\vec{p_1} + \vec{p_2} + \vec{p_3}) = \vec{0},$$

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• Finally, as $\vec{p_i} = k\vec{v}_i + y_i\vec{u}_i$ and $\vec{v}_1 + \vec{v}_2 + \vec{v}_2 = \vec{0}$, we have that

$$y_1\vec{u}_1 + y_2\vec{u}_2 + y_3\vec{u}_3 = \vec{0}.$$

This further implies that $y_1 = y_2 = y_3$.

Conclusion:

- p_i are of equal length and at angles $120°$ to each other,
- $\hat{f}(p_i)$ are of equal length and at angles $120°$ to each other.

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But: Periodicity vectors $\vec{p}$ and $\hat{f}(\vec{p})$ are between vertices of the same triangle lattice. In such a lattice the distances cannot have ratio $\sqrt{8/3}$.

Theorem. The hat tile does not admit a two-way periodic tiling.