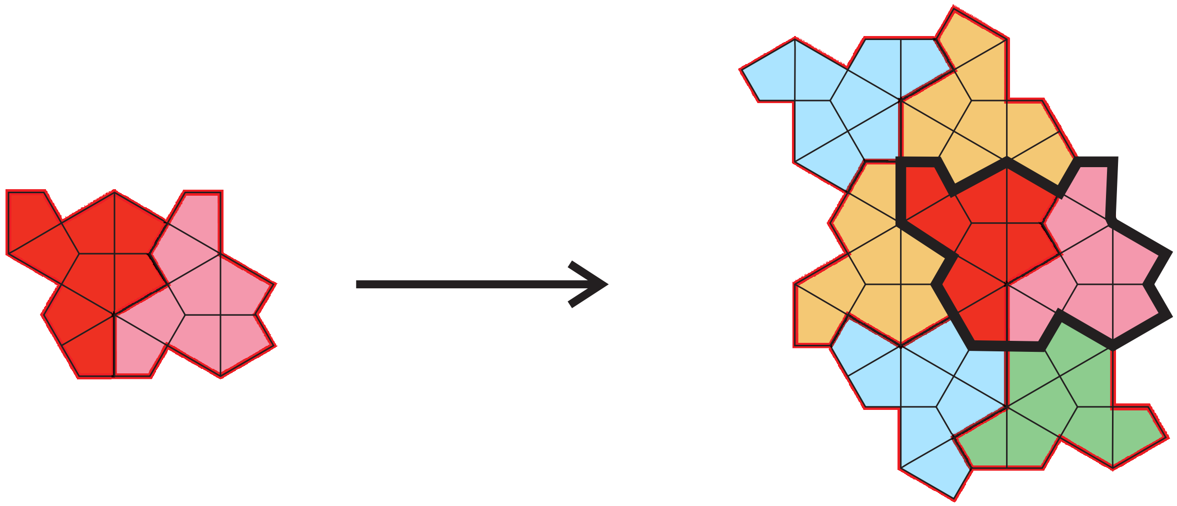
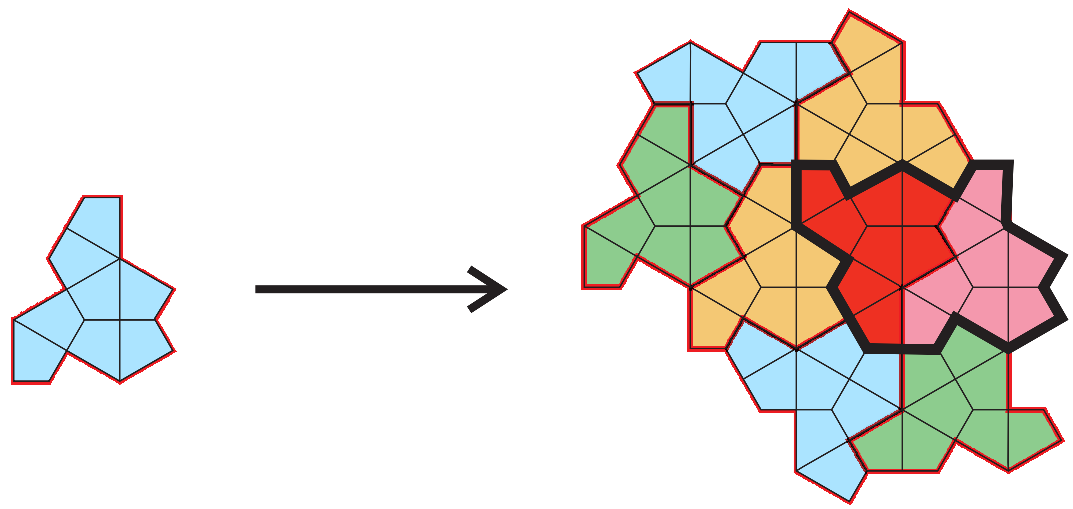
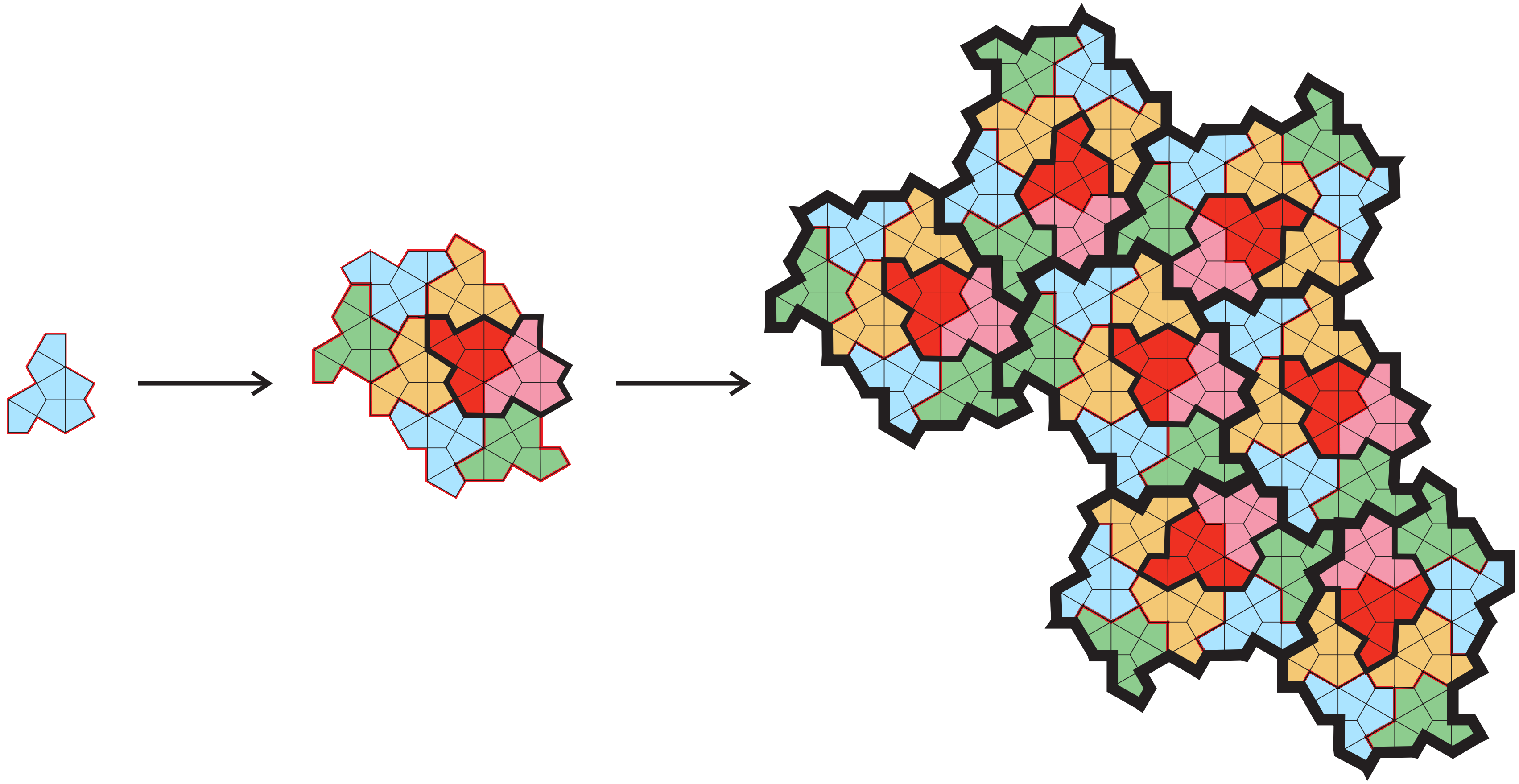
A valid tiling can be generated using a substitution:





First two iterations:



Deformed tiles are also aperiodic monotiles:

Deforming the hat with scaling factors $a \geq 0$ and $b \geq 0$ produces **equivalent** tiles that are also aperiodic, except when

(i) $a = 0$, or

(ii) $b = 0$, or

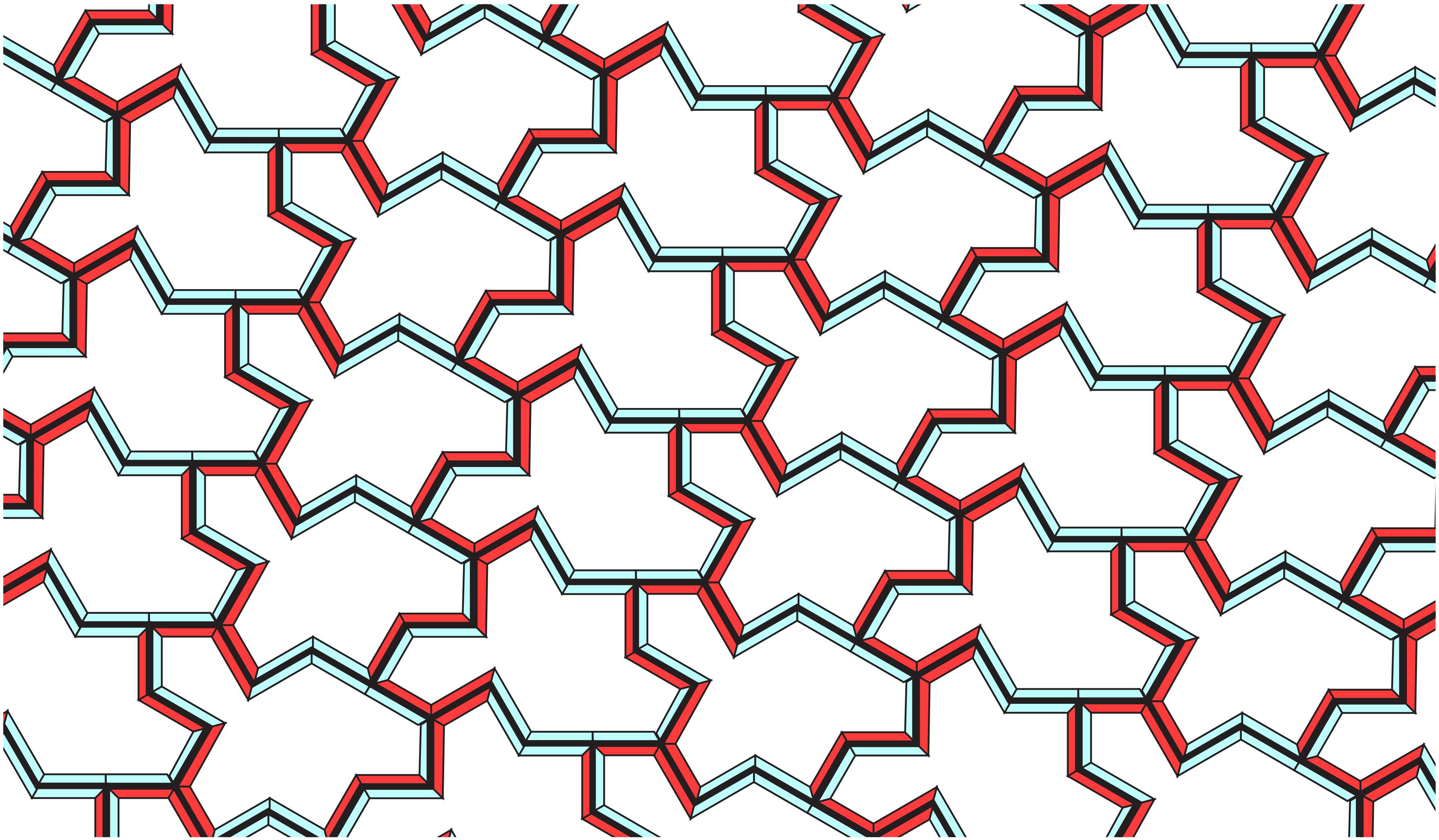
(iii) $b/a = \sqrt{3}$.

In the cases (i) and (ii) the long and short edges vanish, respectively. In the case (iii) the long and short edges become equally long.

• In all cases deforming a tiling by the hat becomes a tiling by the deformed tile.
$\Longrightarrow$ all deformed tiles admit monohedral tilings.

• Conversely, in all cases except (i), (ii) and (iii) in a valid tiling by the deformed tiles long edges meet long edges and short edges meet short edges. Then the tiling can be **inversely deformed** back to a tiling by the hat.
$\Longrightarrow$ the deformed tile does not admit a periodic tiling.

A periodic tiling when the long edges (red) and the short edges (blue) are equally long:



In the periodic tiling both even and odd orientations of the tile are used.

If only even orientations are allowed then the tile still tiles the plane but only non-periodically!

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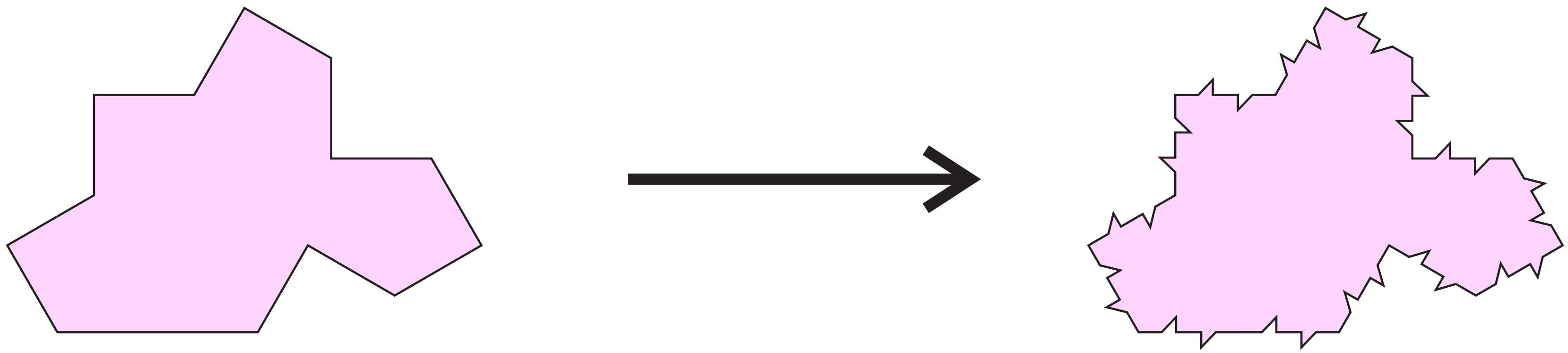
If only even orientations are allowed then the tile still tiles the plane but only non-periodically!

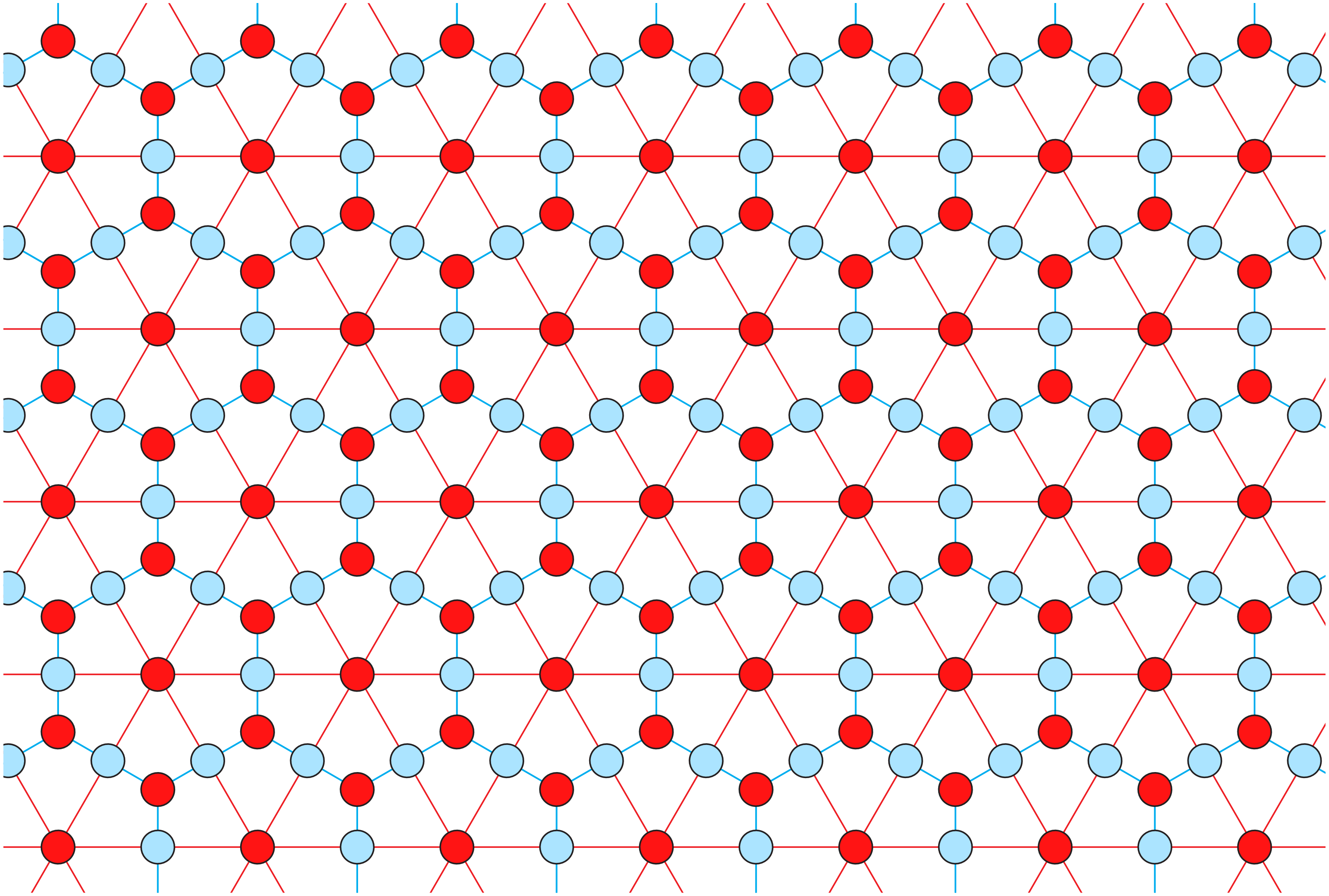
A bumps and dents construction can be used to enforce all tile orientation to have the same parity:



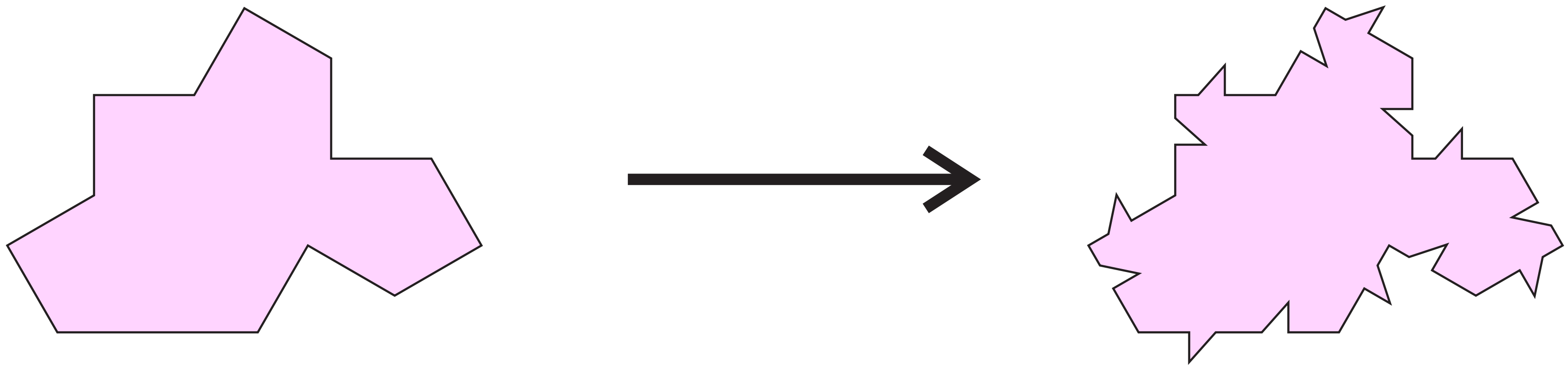
With this tile (**spectre**) then

- there exists a tiling,
- no tiling involves both even and odd variants of the tile,
- there is no valid periodic tiling.

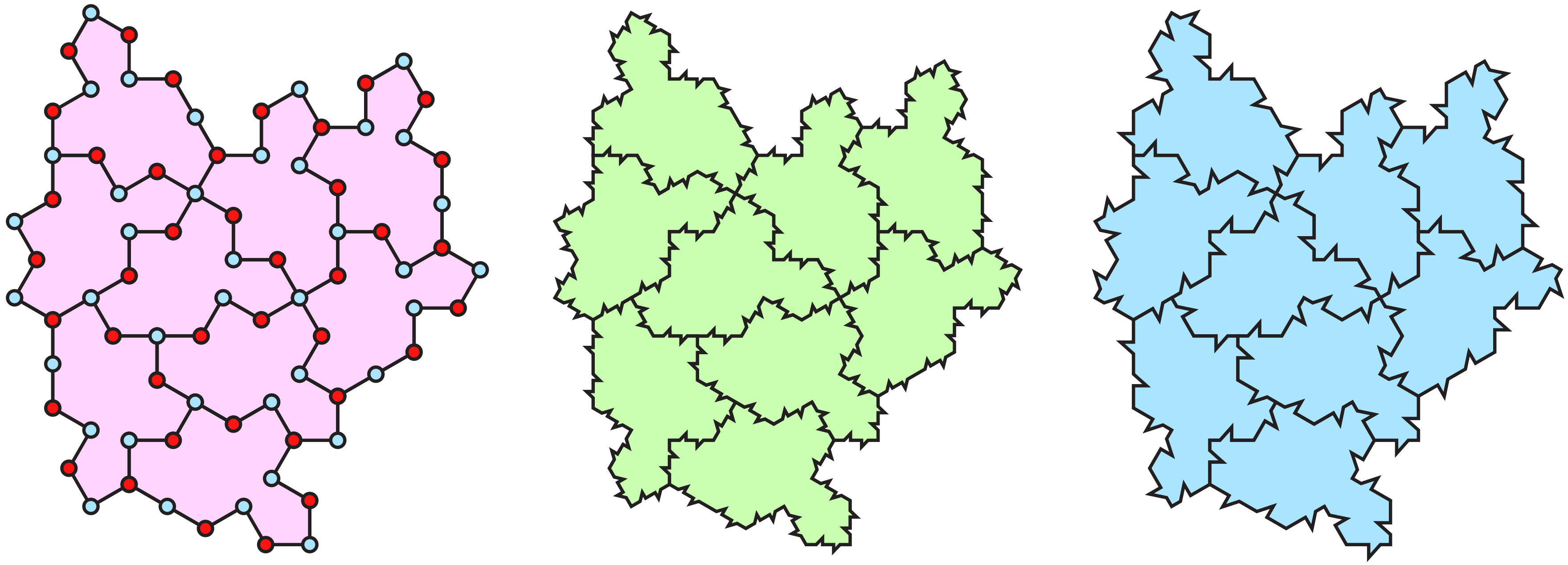
The grid of kites is bipartatite:



This allows to create a **spectre** with simpler bump/dent construction:



A patch tiled with the deformed kite (enforcing all tiles to have even orientations, bipartite coloring shown), and with the two types of **spectres**:



Open problems

Hat is a 13-gon.

Question. What is the smallest n such that there exists an aperiodic n-gon ?
Does there exist an aperiodic pentagon ?

Open problems

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Question. What is the smallest n such that there exists an aperiodic n-gon ? Does there exist an aperiodic pentagon ?

Remark. An aperiodic n-gon cannot be convex: any convex polygon that tiles the plane tiles also periodically.

- Clear for $n \leq 4$ since all triangles and quadrilaterals tile the plane periodically.
- No convex n-gon with $n \geq 7$ can tile the plane. (Use Euler's formula.)
- $n = 6$: Convex hexagons were analysed by K. Reinhardt in 1918: there are only three types of convex hexagons that tile the plane and they all tile periodically.
- $n = 5$: Classifying convex pentagonal monotilings was completed in 2017: there are 15 types of tiles and they all tile periodically. (The last of the 15 types was discovered as late as in 2015!)

The conversion

$$\text{Wang tiles} \longrightarrow \text{polygons}$$

with bumps and dents is effective (=algorithmic). The vertices of the resulting polygons can be taken to have rational number coordinates.

So, the undecidability results proved for Wang tiles hold for polygonal prototiles as well:

Theorem. The following decision problems are undecidable:

- "Does a given protoset of polygons with rational coordinates admit a periodic tiling?",
- "Does a given protoset of polygons with rational coordinates admit a tiling?".

Proof.

It is not known if there exists a decision algorithm to determine if a given **single polygonal prototile** admits a valid (periodic) tiling.

Question. Are the following decision problems decidable ?

- "Does given single polygon with rational coordinates admit a tiling?"
- "Does given single polygon with rational coordinates admit a periodic tiling?"

One may also consider similar questions for **polyominoes** (=tiles that are edge-to-edge attachments of unit squares to each other.)

Theorem. The tiling problem is undecidable for protosets of 5 polyominoes.

In particular, this also implies the undecidability of the tiling problem among sets of 5 polygons.

On the other hand, the tiling problem is known to be decidable for single polyominoes if only translations are allowed, that is, the tiles must be placed in the given orientation.

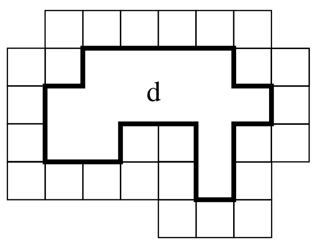
Here is a related question by H. Heesch. Consider a single prototile t that does not admit a tiling of the plane.

The **Heesch number** of t is the maximum number of times the tile can be completely surrounded by copies of t. More precisely, for a topological disk $d \subseteq \mathbb{R}^2$, a **corona** of d is a collection C of tiles, all congruent to t, such that

- the interiors of the elements of C are pairwise disjoint, and disjoint from d, and
- $d \cup \bigcup_{s \in C} s$ is a topological disk whose interior contains d.

In other words, tiles in the corona C surround set d completely.

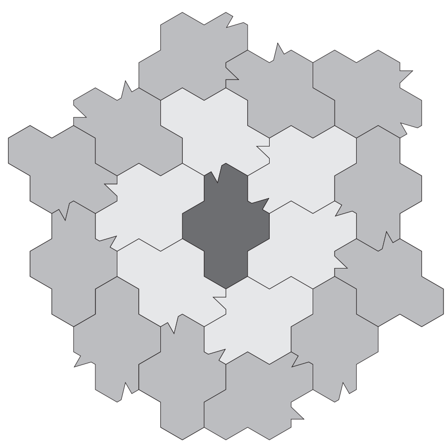
Example. The squares form a corona of the set d in the middle:



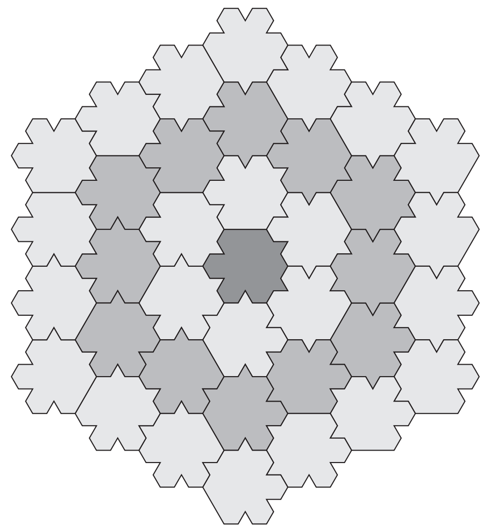
A a second corona of d is a corona of the set that is the union of d and its first corona. Inductively, a $k+1$'st corona is a corona of the topological disk formed by d and its first k coronas.

In the Heesch problem we start with a single copy of t and form its 1st, 2nd, 3rd, etc. coronas. If the k'th corona exists for every k then by the extension theorem the entire plane can be tiled. But if t does not admit a plane tiling then there exists the largest k such that the first k coronas exist. This k is called the **Heesch number** of tile t.

Example. The following figure illustrates two coronas of a tile:



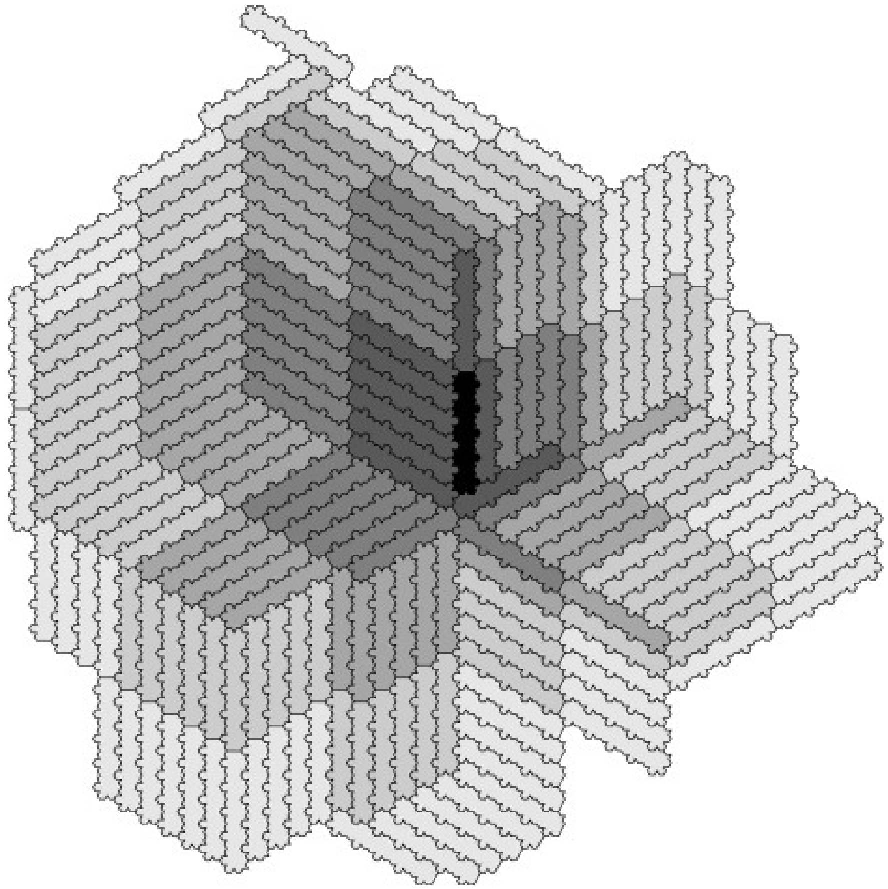
Example. A regular hexagon with incoming arrows on three sides and outgoing arrows on two sides admits three coronas:



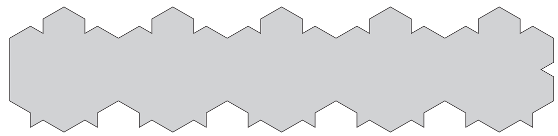
(In the picture, the arrows are represented by bumps and dents.)

Due to the imbalance in the number of incoming and outgoing arrows the full plane cannot be tiled by this tile. (Similar to a prof at the homeworks for Wang tiles.)

Example. Heesch number five (by Casey Mann):

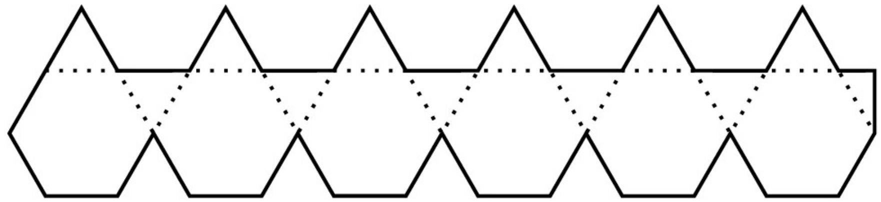


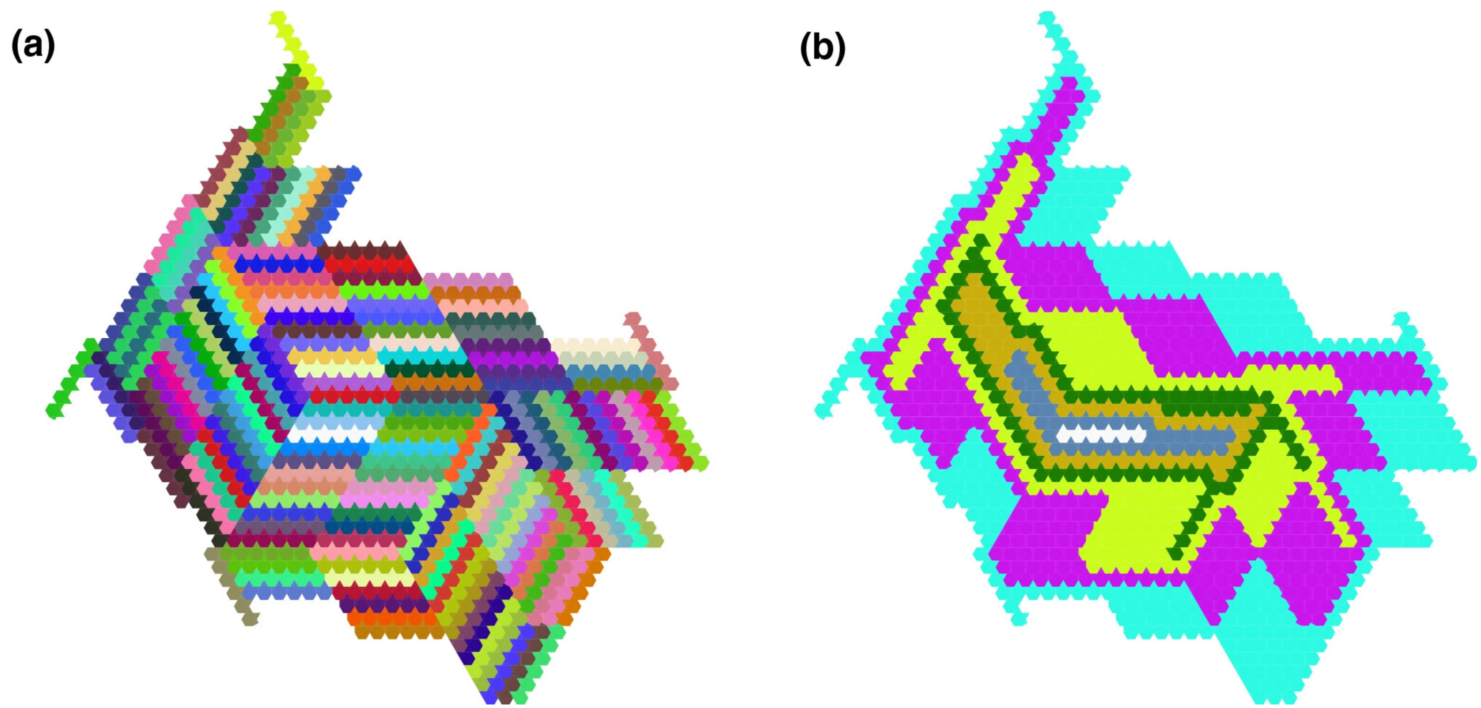
The tile consists of five regular haxagons glued together, with bumps and dents on 10 and 11 sides:



The imbalance in the number of bumps and dents guarantees that no valid tiling of the plane is possible.

Example. Recently, a tile with Heesch number 6 was published:





Heesch's question. Does there exist a number k such that the Heesch number of every tile that does not admit a tiling is at most k ? If such a k exists, what is the smallest such k ?

Note that if the Heesch numbers are bounded by some constant k then there is an **algorithm** (in any reasonable set-up such as edge-to-edge tilings by polygons where one can try all possible coronas) to determine if a given single tile admits a tiling: To test if a tiling exists, all we need to do is to try all possible ways of building $k+1$ coronas. A valid tiling exists if and only if $k+1$ coronas exist.