Cellular Automata. Homework 2 (2.2.2026)

1. In *Game-of-Life*, find a finite configuration c such that $G(c)$ has exactly one living cell. Find also a configuration e such that in $G(e)$ all cells are alive.

2. Let $c \in S^{\mathbb{Z}^d}$ be a configuration that is periodic in d linearly independent directions $\vec{v}_1, \ldots, \vec{v}_d$. Let $\vec{u} \in \mathbb{Z}^d$ be an arbitrary integer vector. Prove that there exists integer $k > 0$ such that c is $k\vec{u}$-periodic.

3. Prove that if the orbit $c, G(c), G^2(c), \ldots$ is a converging sequence of configurations then its limit is a fixed point of G.

4. Let G be the majority rule from last week's exercise 6.

   (a) Determine all fixed points of G,

   (b) Determine which configurations c have a converging orbit $c, G(c), G^2(c), \ldots$.

5. (Parallel converging subsequences) Let K be a set, and for each $k \in K$ let

$$c_1^{(k)}, c_2^{(k)}, c_3^{(k)}, \ldots$$

   be a sequence of elements of $S_k^{\mathbb{Z}^d}$ where S_k is a finite set.

   (a) Prove that if K is countable then there exists a sequence $i_1 < i_2 < i_3 < \ldots$ of indices such that the subsequence

$$c_{i_1}^{(k)}, c_{i_2}^{(k)}, c_{i_3}^{(k)}, \ldots$$

   converges for every $k \in K$.

   (b) Give an example of sequences (with uncountable K) such that for every choice $i_1 < i_2 < i_3 < \ldots$ of indices some of the subsequences

$$c_{i_1}^{(k)}, c_{i_2}^{(k)}, c_{i_3}^{(k)}, \ldots$$

   does not converge.

6. Prove the Garden-Of-Eden -inequality: For all $d, n, s, r \in \mathbb{Z}_+$, the following inequality holds for all sufficiently large $k \in \mathbb{Z}_+$:

$$\left(s^{n^d} - 1\right)^{k^d} < s^{(kn-2r)^d}.$$

7. Let $A = (d, \{0,1\}, M_1^2, f)$ be the 2-dimensional, two-state CA with the Moore neighborhood whose local rule f is the majority rule:

$$f(a,b,c,d,e,f,g,h,i) = \begin{cases} 0, & \text{if } a+b+c+d+e+f+g+h+i \leq 4, \\ 1, & \text{if } a+b+c+d+e+f+g+h+i \geq 5. \end{cases}$$

   Find an *orphan* for this CA, that is, find a finite pattern $p = (D, g)$ such that any configuration that contains p is a Garden-of-Eden.