Injectivity and surjectivity

Standard notations: Let $g : A \longrightarrow B$ be a function.

- For any $K \subseteq A$ we denote
$$g(K) = \{g(k) \mid k \in K\}.$$
- For any $L \subseteq B$ we denote
$$g^{-1}(L) = \{a \in A \mid g(a) \in L\}.$$
- For $b \in B$ the set
$$g^{-1}(b) = \{a \in A \mid g(a) = b\}$$
is the set of pre-images of element b.

Injectivity and surjectivity

Standard concepts: Function $g : A \longrightarrow B$ is called

- **injective** or **one-to-one** if every element of B has at most one pre-image:

$$|g^{-1}(b)| \leq 1 \text{ for all } b \in B,$$

- **surjective** or **onto** if every element of B has at least one pre-image:

$$|g^{-1}(b)| \geq 1 \text{ for all } b \in B,$$

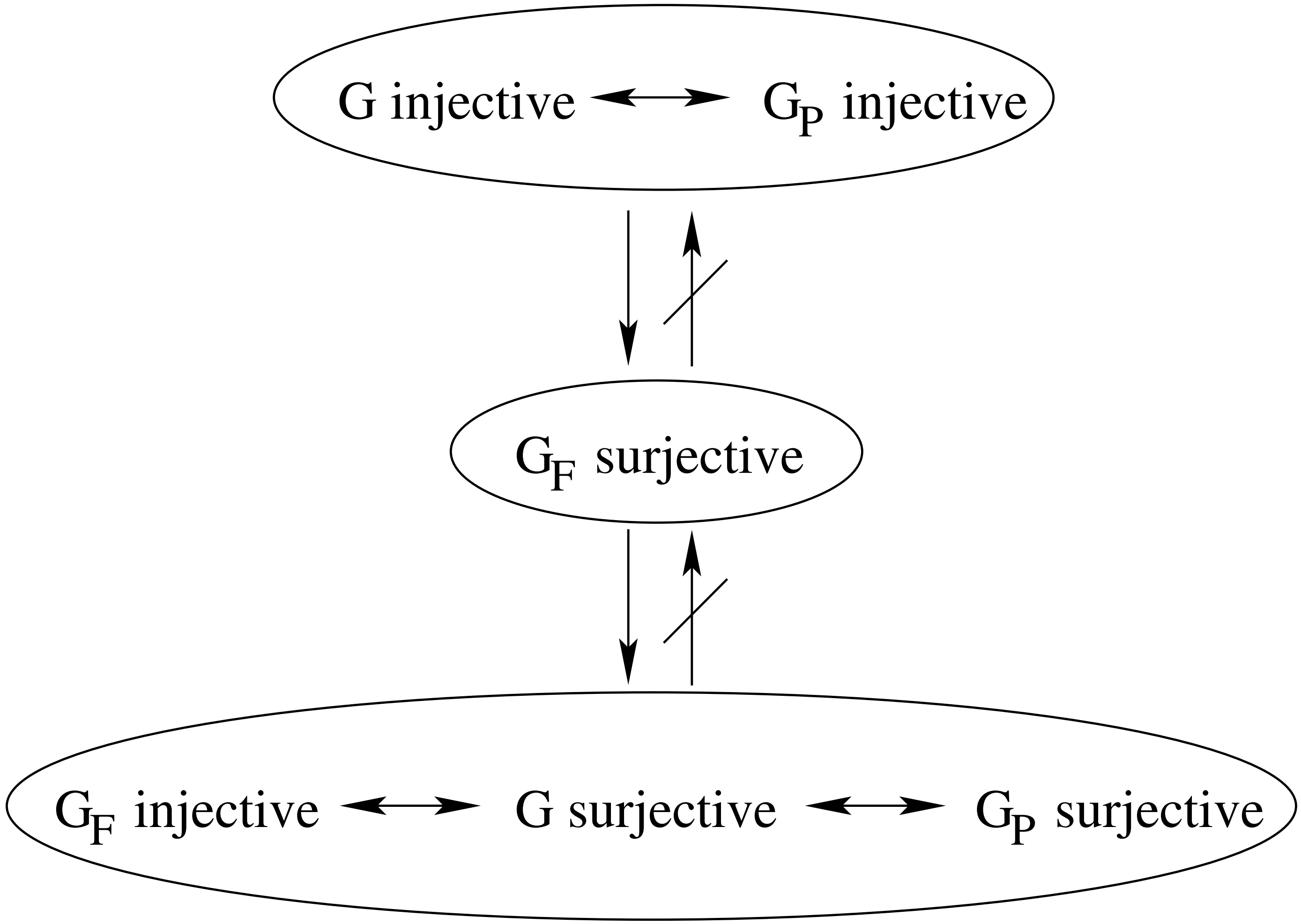
- **bijective** if it is both injective and surjective, i.e., every element of B has exactly one pre-image:

$$|g^{-1}(b)| = 1 \text{ for all } b \in B,$$

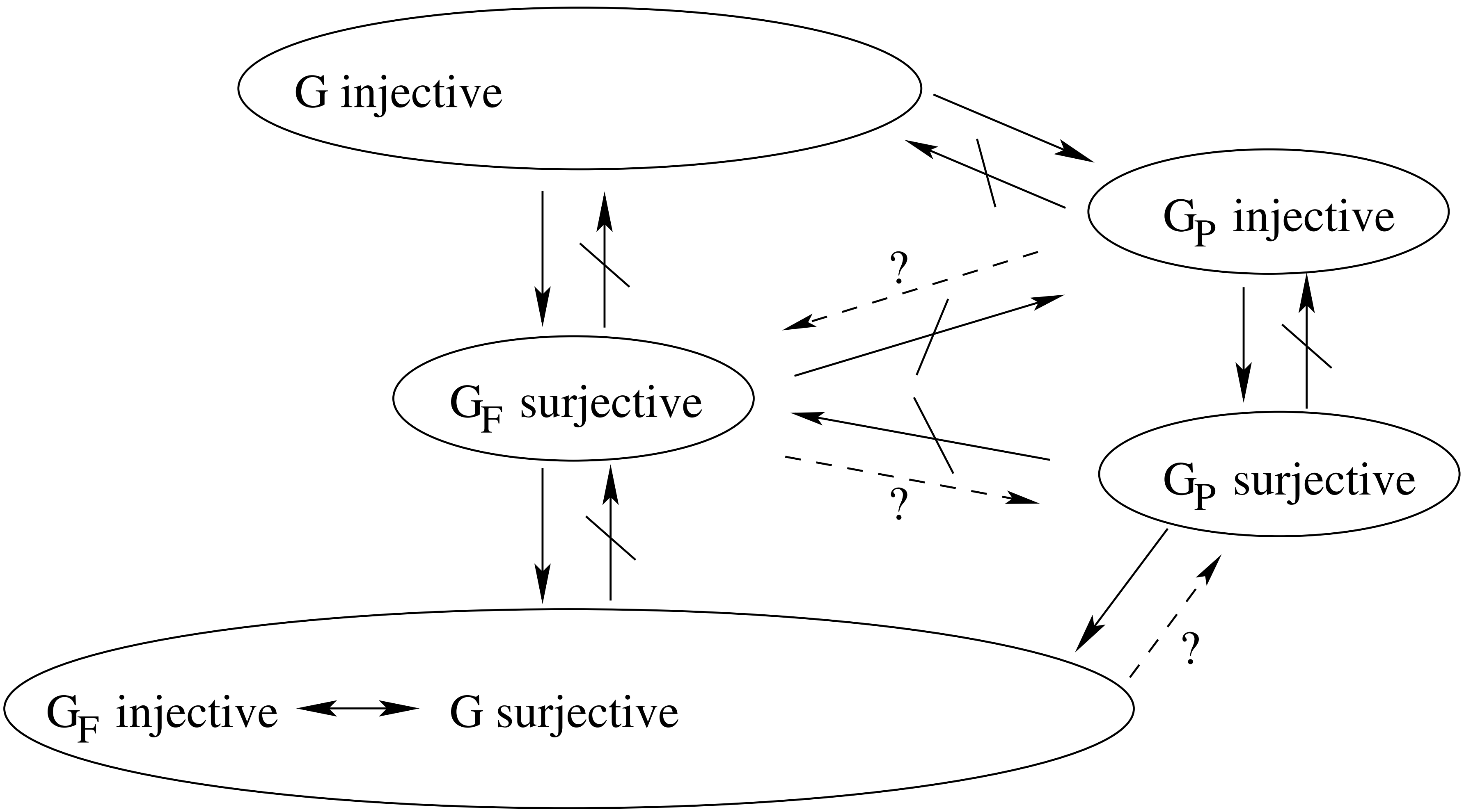
A CA is called **injective**, **surjective** or **bijective** if its transition function G is injective, surjective or bijective, respectively.

There exist several implications between injectivity, surjectivity and bijectivity properties of functions G, G_F and G_P.

In the one-dimensional case $d = 1$:



In the two- and higher dimensional cases $d \geq 2$:



First the simple implications:

Proposition. For any CA function G holds:

(a) If G is injective then also G_F and G_P are injective.

(b) If G_F or G_P is surjective then also G is surjective.

(c) If G_P is injective then G_P is surjective.

Proof.

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Proposition. For any CA function G holds:

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(c) If G_P is injective then G_P is surjective.

Proof.

Corollary. Every injective CA is surjective, so injectivity is equivalent to bijectivity.

Proof. G injective $\implies G_P$ injective $\implies G_P$ surjective $\implies G$ surjective. $\square$

Reversible CA

A CA A and the function G it defines are called **reversible** if

- G is bijective, and
- the inverse function G^{-1} is also a CA function.

We call G^{-1} the **inverse automaton**.

The inverse automaton retraces the orbits backwards in time.

Example. $d = 1$, $S = \{1, 2, 3\}$, $N = (0, 1)$, and the value $f(a, b)$ is given by the following table:

$a \backslash b$	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

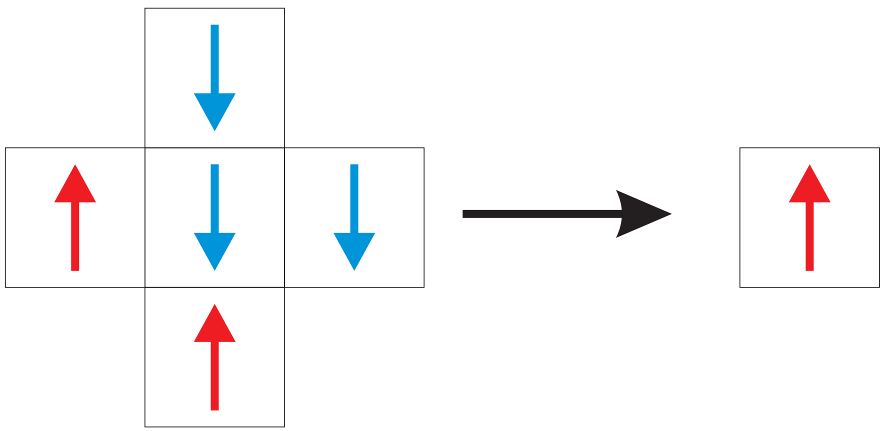
State 3 does not change. Swap $1 \longleftrightarrow 2$ iff the right neighbor is 3.

Is reversible: G^2 is the identity map, so G is its own inverse. It is an **involution**.

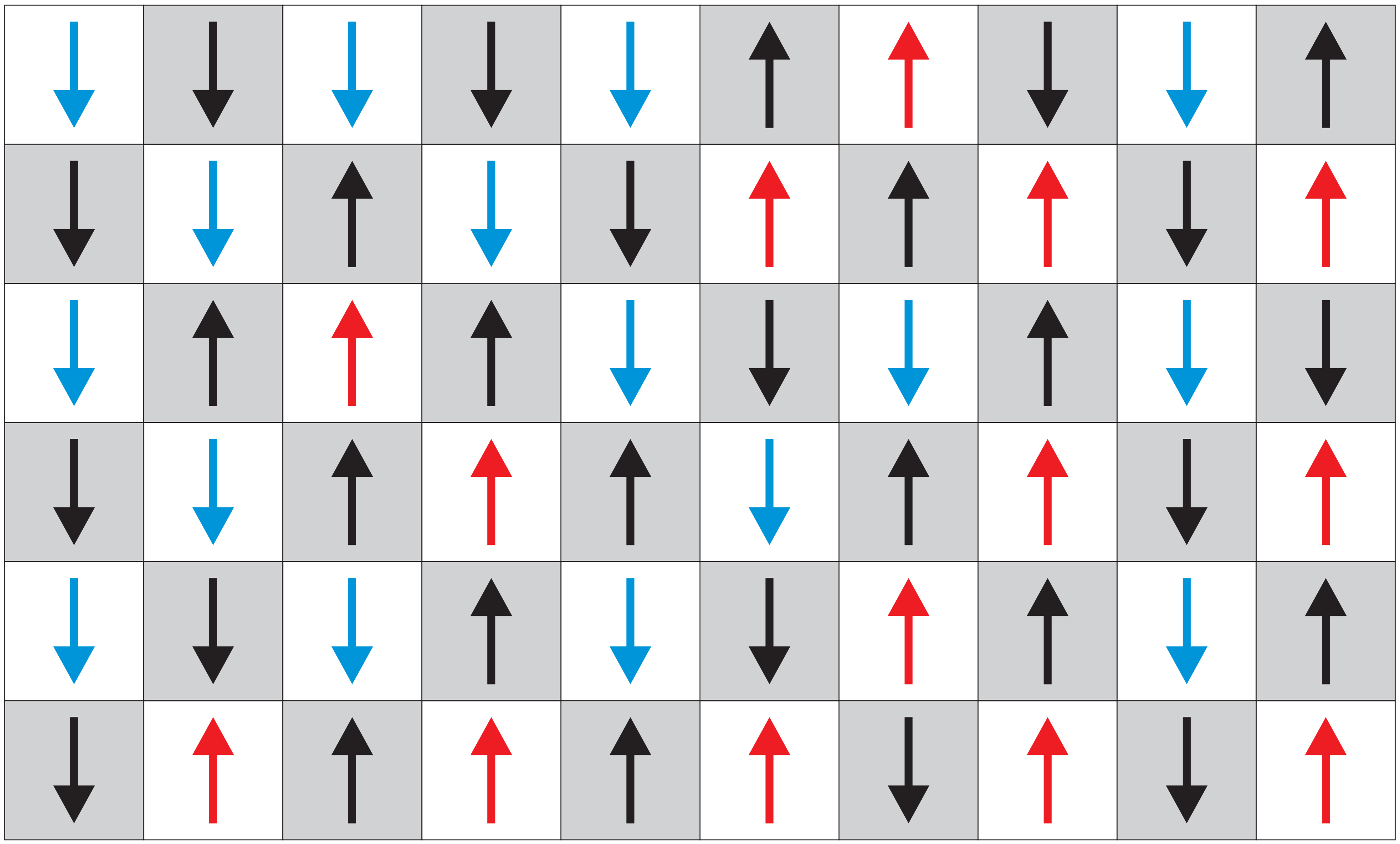
Note how the inverse rule gives $c(n)$ based on $G(c)(n)$ and $G(c)(n+1)$ even though $c(n)$ does not influence $G(c)(n+1)$ in any way in the forward direction.

Example. Two-dimensional **Q2R** Ising model by G.Vichniac (1984).

Each cell has a **spin** that is directed either up or down. The direction of a spin is swapped if and only if among the four immediate neighbors there are exactly two cells with spin up and two cells with spin down:



The twist that makes the Q2R rule reversible: Color the space as a checkerboard. On even time steps only update the spins of the white cells and on odd time steps update the spins of the black cells.



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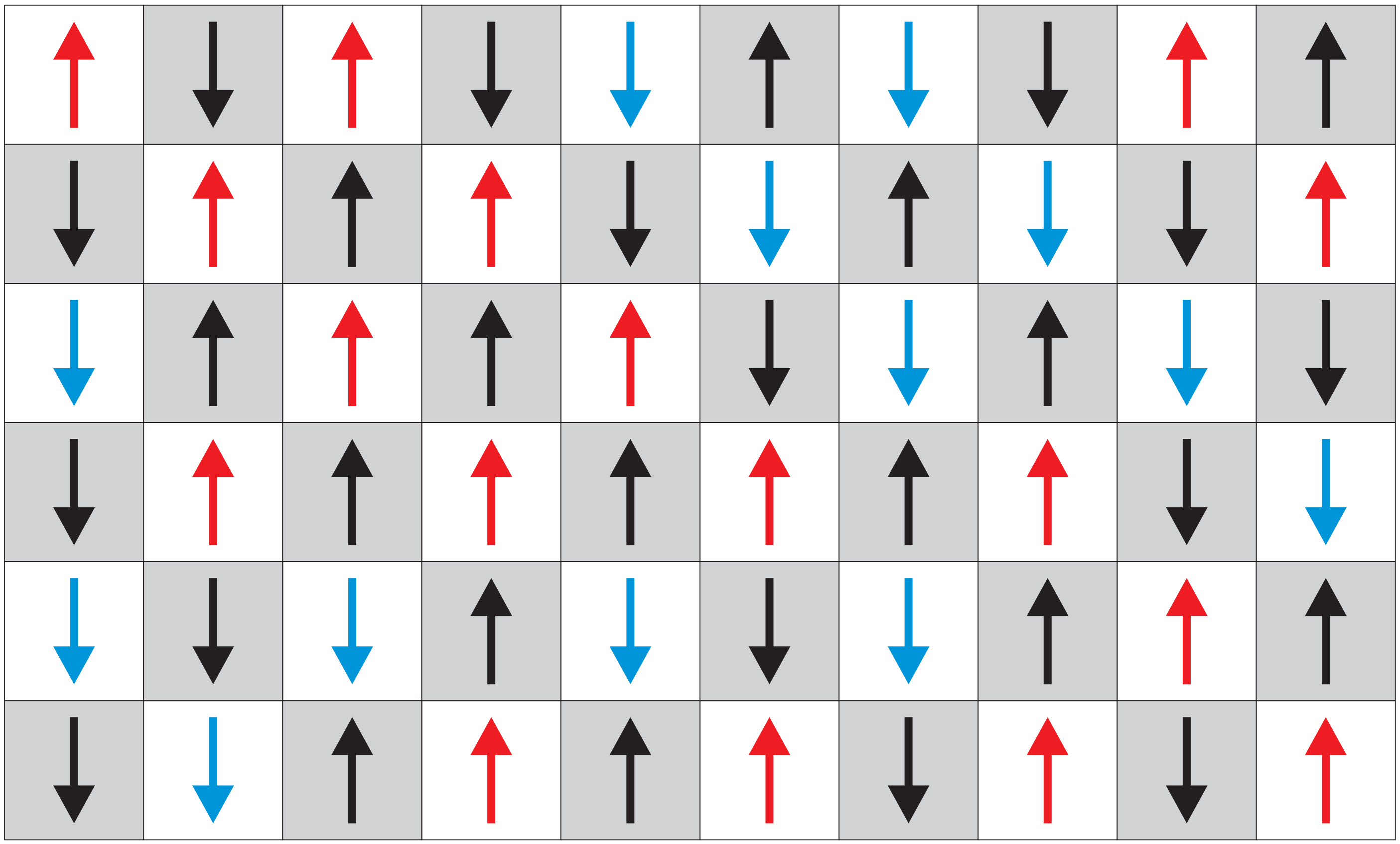

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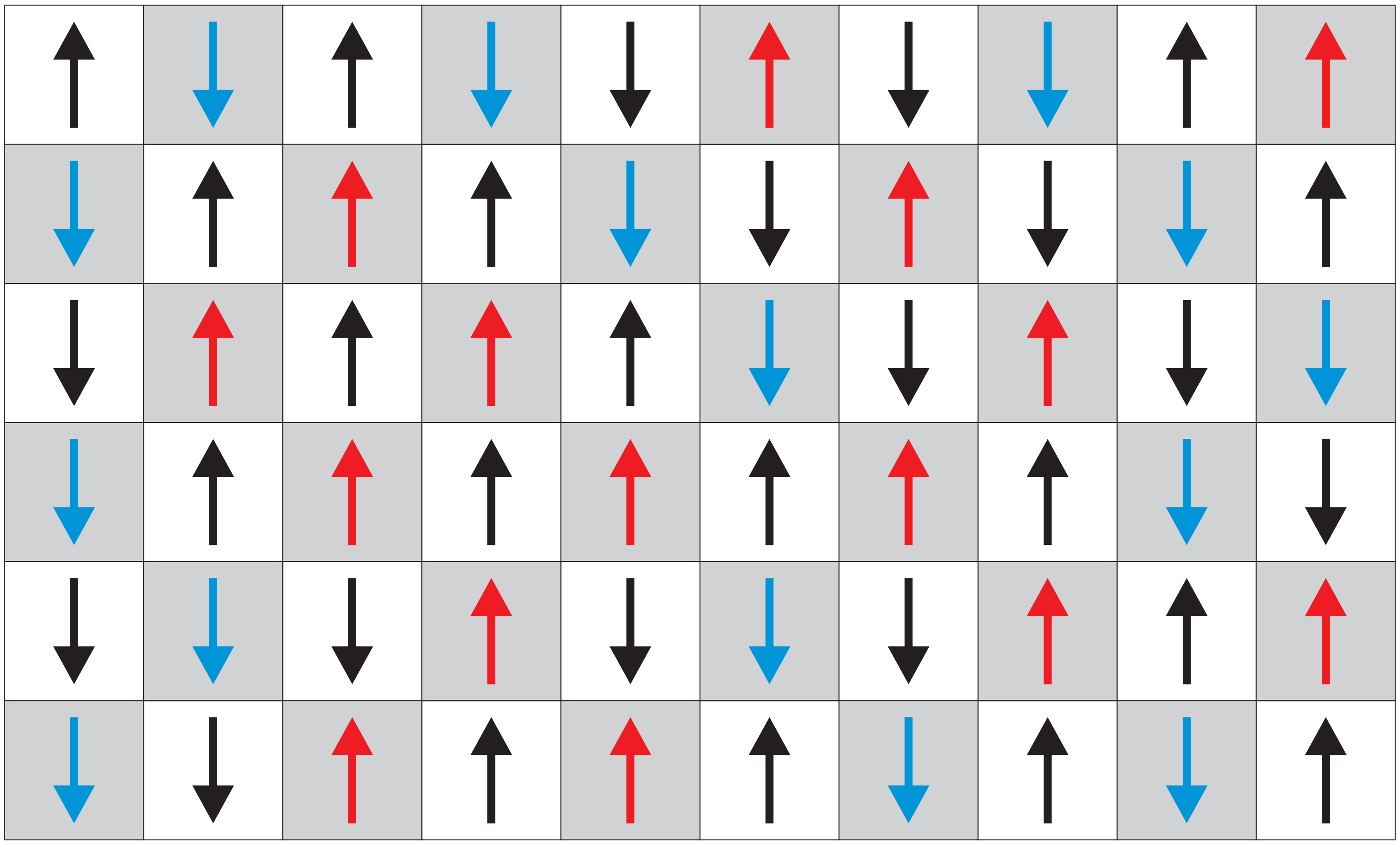

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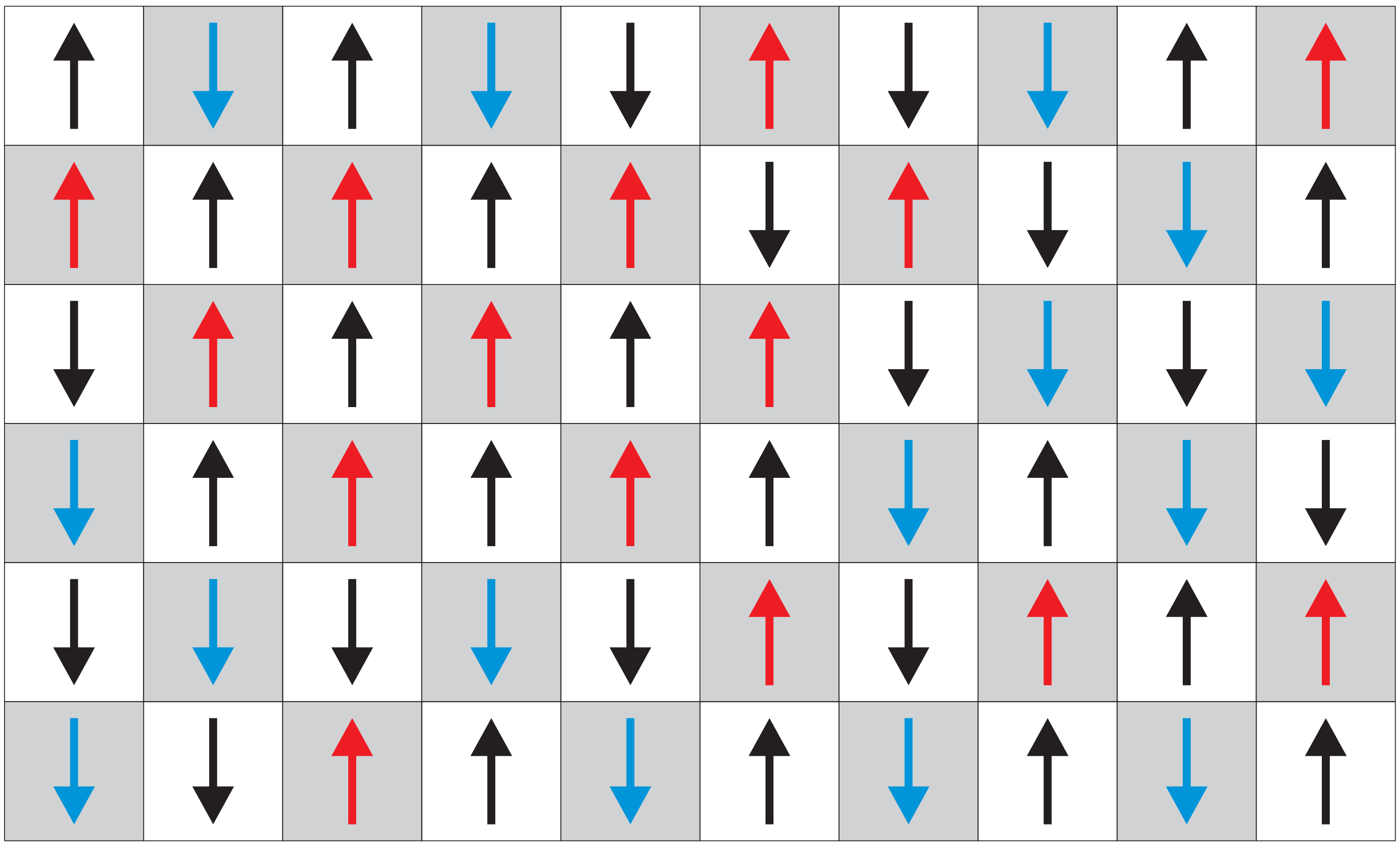
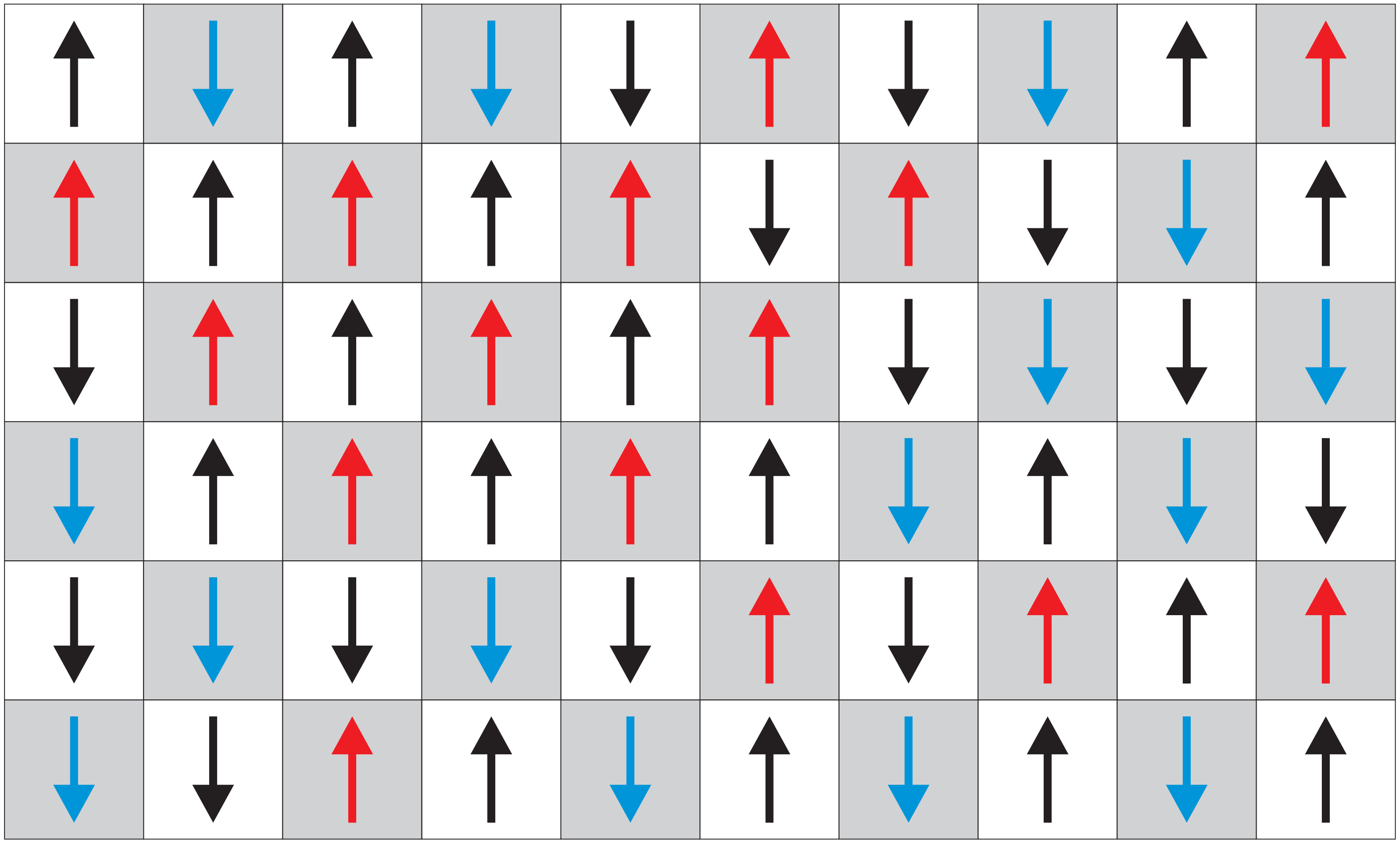
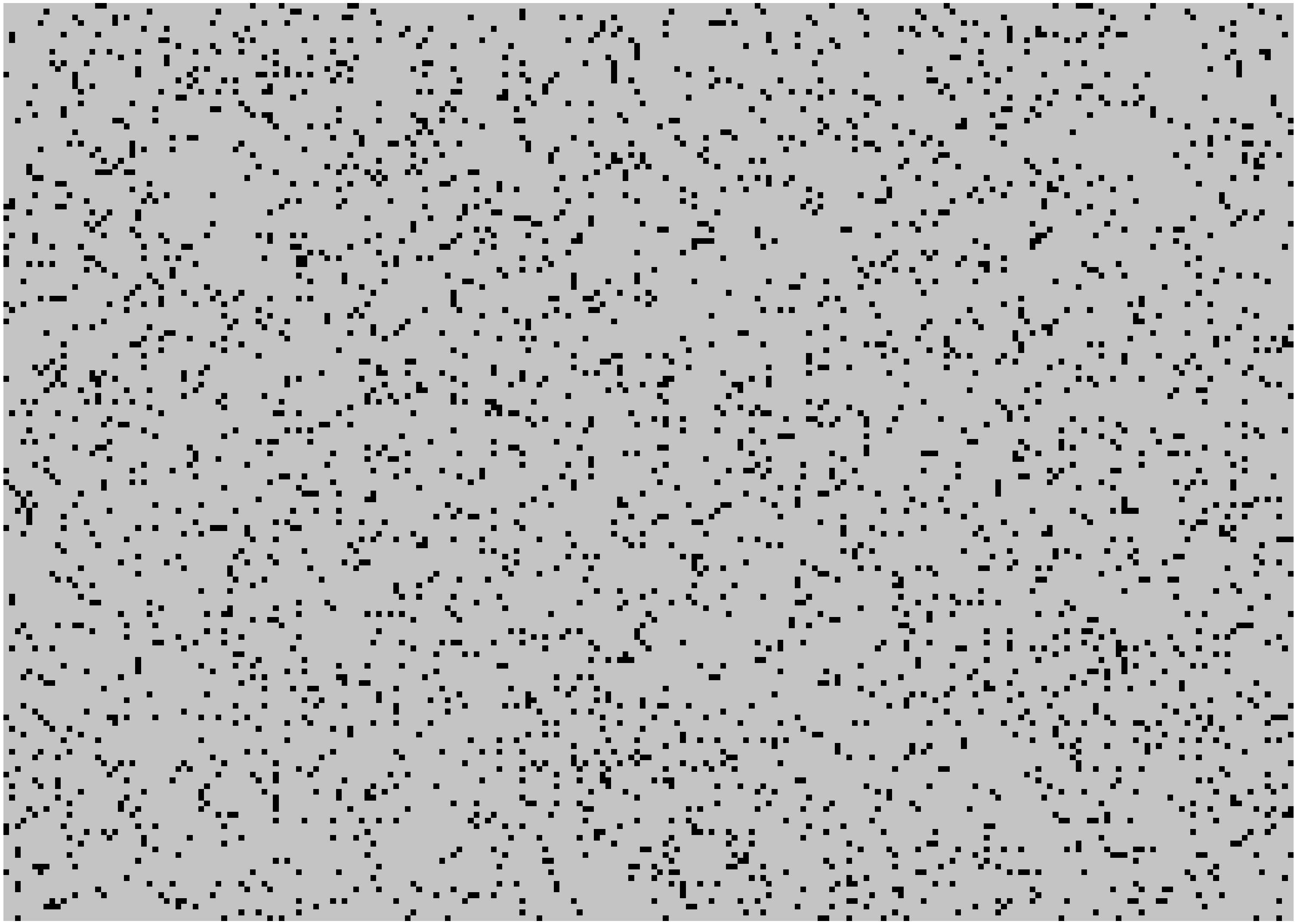


Q2R is **reversible**: The same rule (applied again on squares of the same color) reconstructs the previous generation.

Q2R rule also exhibits a local **conservation law**: The number of neighbors with opposite spins remains constant over time.

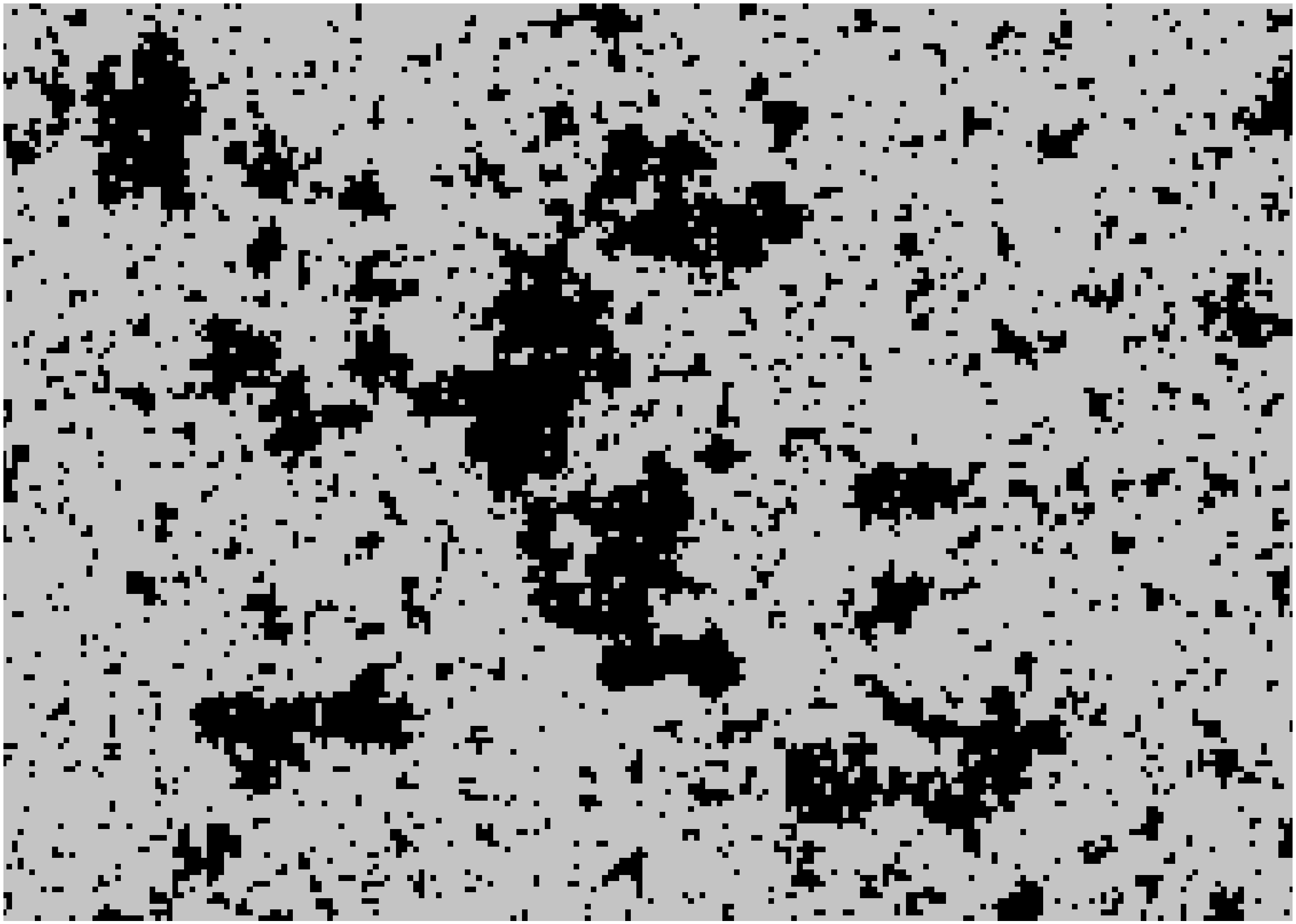


Evolution of Q2R from an uneven random distribution of spins:



Initial random configuration with 8% spins up.

Evolution of Q2R from an uneven random distribution of spins:



After approx. one million steps. Notice the clustering.

Every reversible CA has to be bijective by definition. The converse is also true: If G is bijective then every cell only needs to know states of finitely many neighbors in $G(c)$ to determine its state in c.

Proposition. Every bijective CA is reversible.

Proof. Based on compactness.

Because injectivity implies surjectivity we have:

$$G \text{ is injective} \iff G \text{ is bijective} \iff G \text{ is reversible}$$

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Corollary. If G is injective then G_F is surjective.

Proof.