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It turns out that all surjective CA have **balanced** local rules f: for all $a \in S$

$$\left|f^{-1}(a)\right| = |S|^{m-1}$$

where S is the state set and m is the size of the neighborhood.

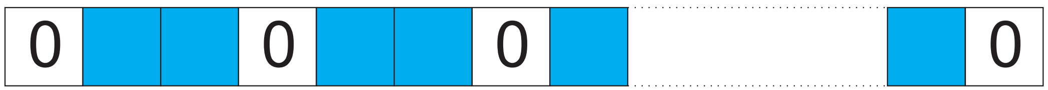
The following example indicates how to prove this.

Example. Consider a non-balanced local rule such as rule 110 where five contexts give new state 1 while only three contexts give state 0:

$$\begin{array}{lcl} 111 & \longrightarrow & 0 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 1 \\ 010 & \longrightarrow & 1 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

Let us show how this non-balance implies that 110 is not surjective.

Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where k is the number of 0's.



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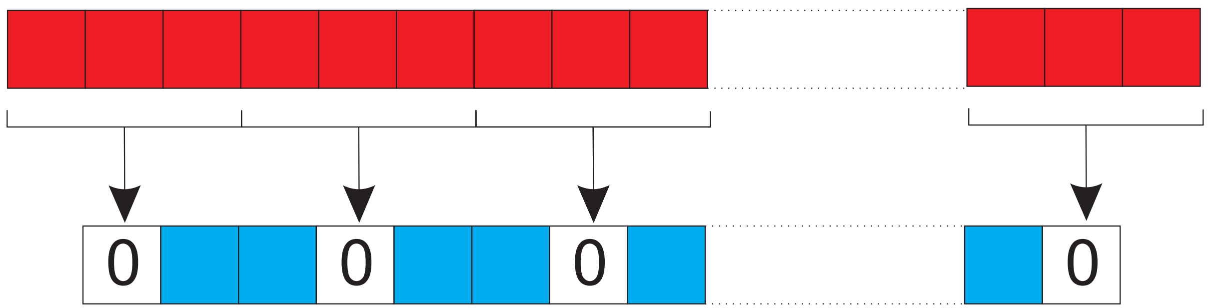


A pre-image of such a pattern must consist of k segments of length three, each of which is mapped to 0 by the local rule. There are 3^k choices.

As for large values of k we have $3^k < 4^{k-1}$, there are fewer choices for the red cells than for the blue ones. Hence some pattern has no pre-image and therefore the CA is not surjective $\square$

Let us generalize the previous example.

A **pattern**

$$p = (D, g)$$

is a partial configuration with domain $D \subseteq \mathbb{Z}^d$ and assignment $g : D \longrightarrow S$ of states to the domain.

We also denote $g \in S^D$.

The pattern is **finite** if D is a finite set.

If $\tau = \tau_{\vec{r}}$ is a translation of $\mathbb{Z}^d$ then the **translated** pattern $\tau(p)$ is defined in the obvious manner: its domain is $D - \vec{r}$ and the state in $\vec{n} \in D - \vec{r}$ is $g(\vec{n} + \vec{r})$.

A pattern $p_1 = (D_1, g_1)$ is a **subpattern** of $p_2 = (D_2, g_2)$ if $D_1 \subseteq D_2$ and $g_1|_{D_1} = g_2|_{D_1}$.

Patterns p_1 and p_2 are **disjoint** if $D_1 \cap D_2 = \emptyset$.

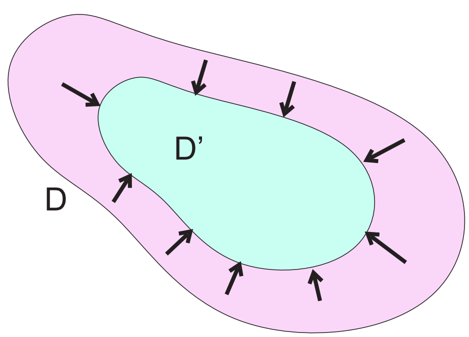
Let G be CA function defined by CA $A = (d, S, N, f)$ where

$$N = (\vec{n}_1, \ldots, \vec{n}_m).$$

Let $p = (D, g)$ be a pattern, and let $D' \subseteq \mathbb{Z}^d$ a domain such that $N(D') \subseteq D$, so all neighbors of all cells in D' are in D.

The local rule maps the pattern $p = (D, g)$ in a natural manner into the pattern $p' = (D', g')$ where

$$\forall \vec{n} \in D' \; : \; g'(\vec{n}) = f[g(\vec{n} + \vec{n}_1), g(\vec{n} + \vec{n}_2), \ldots, g(\vec{n} + \vec{n}_m)].$$



The mapping $p \mapsto p'$ is denoted by

$$\mathbf{G}^{(\mathbf{D} \to \mathbf{D}')}$$

(or simply by G when the domains D and D' are clear from the context.)

A finite pattern without a pre-image is called an **orphan**.

So $p' = (D', g')$ is an orphan if $G^{(D \to D')}(p) \neq p'$ for all $p = (D, g)$ with domain $D = N(D')$.

Clearly any configuration that contains a copy of an orphan is a Garden-of-Eden configuration. Also the converse is true:

Proposition. Every Garden-of-Eden configuration has a subpattern that is an orphan. Hence, a cellular automaton is non-surjective if and only if there exists an orphan.

Proof.

A technical result (Garden-Of-Eden inequality):

Lemma. For all $d, n, s, r \in \mathbb{Z}_+$ there exists $k \in \mathbb{Z}_+$ such that

$$\left(s^{n^d} - 1\right)^{k^d} \; < \; s^{(kn-2r)^d}.$$

Proof. In the homework assignments.

Proposition (the balance property of surjective CA). Let

$$A = (d, S, N, f)$$

be a surjective CA, and let $D, D' \subseteq \mathbb{Z}^d$ be finite domains such that $N(D') \subseteq D$. For every pattern $p' = (D', g')$ the number of patterns $p = (D, g)$ such that

$$G^{(D \to D')}(p) = p'$$

is $s^{|D|-|D'|}$ where $s = |S|$ is the number of states.

Proof.

The balance property states that surjective CA preserve the **uniform probability distribution** of configurations.

As will be discussed later in the class, uniform randomness means that for every finite domain $D \subseteq \mathbb{Z}^d$, all finite patterns on domain D have the same probability $1/s^{|D|}$.

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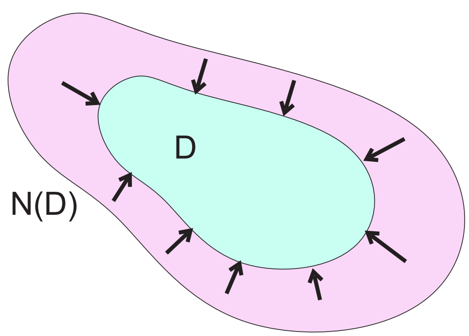
By the balance property of a surjective G, if c is drawn uniformly randomly then also $G(c)$ is distributed uniformly randomly: For each D, all finite patterns with domain D are obtained by G from the same number of patterns with domain $N(D)$. Since all these patterns have equal probabilities of being in c, all patterns with domain D have equal probabilities of appearing in $G(c)$.



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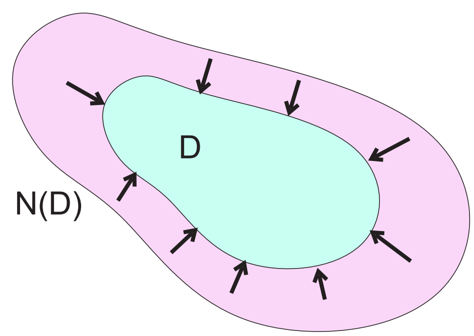

So starting a surjective CA from uniformly random initial configuration will produce nothing but white noise!

A special case: the local rule table of a surjective cellular automaton is balanced.

Corollary. In surjective CA

$$|f^{-1}(a)| = |S|^{m-1}$$

for all $a \in S$, where m is the size of the neighborhood and

$$f^{-1}(a) = \{(s_1, \dots, s_m) \mid f(s_1, \dots, s_m) = a\}.$$

Proof. Balance property with $D' = \{\vec{0}\}$. $\square$

Example. Balance in the local rule is not sufficient to guarantee surjectivity.

Elementary CA 232 (the **majority CA**) has a balanced rule table:

$$\begin{array}{lcl} 111 & \longrightarrow & 1 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 1 \\ 010 & \longrightarrow & 0 \\ 001 & \longrightarrow & 0 \\ 000 & \longrightarrow & 0 \end{array}$$

However, the balance fails on patterns of length two: Any word of length four that contains at most one state 1 is mapped to 00, and so 00 has at least 5 pre-images of length four. (The balanced number of pre-images would be 4.)

So the **majority CA** is not surjective.