Garden-of-Eden -theorem

One of the oldest results in CA theory:

$$\mathbf{G} \text{ is surjective} \iff \mathbf{G_F} \text{ is injective}$$

($\Longrightarrow$ by E.F.Moore in 1962, and $\Longleftarrow$ by J.Myhill in 1963)

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Injectivity of G_F requires a quiescent state. A more robust concept is pre-injectivity:

Configurations c_1 and c_2 are **asymptotic** if the set

$$\mathit{diff}\,(c_1, c_2) = \{\vec{n} \in \mathbb{Z}^d \mid c_1(\vec{n}) \neq c_2(\vec{n})\ \}$$

of positions where c_1 and c_2 differ is finite.

Cellular automaton G is **pre-injective** if for any asymptotic c_1 and c_2 holds

$$c_1 \neq c_2 \Longrightarrow G(c_1) \neq G(c_2).$$

Clearly all injective CA are pre-injective.

For pre-injectivity it is enough that the CA is one-to-one among c-asymptotic configurations, for any fixed configuration c.

Proposition. Let $c \in S^{\mathbb{Z}^d}$ be arbitrary. Cellular automaton G is pre-injective if and only if it is injective on

$$asymp(c) = \{e \in S^{\mathbb{Z}^d} \mid \ c \text{ and } e \text{ are asymptotic } \}.$$

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In particular: G (with a quiescent state q) is pre-injective iff G_F is injective. (Apply the proposition with $c = q-$uniform configuration.)

So the Garden-of-Eden -theorem states that

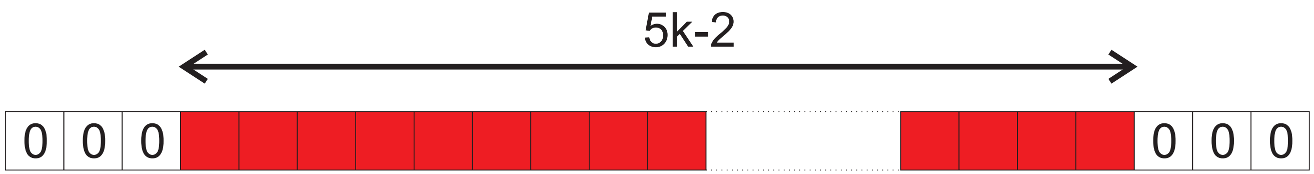
$$\mathbf{G} \text{ is surjective} \iff \mathbf{G} \text{ is pre-injective}$$

Myhill direction: G not surjective $\Longrightarrow$ G not pre-injective.

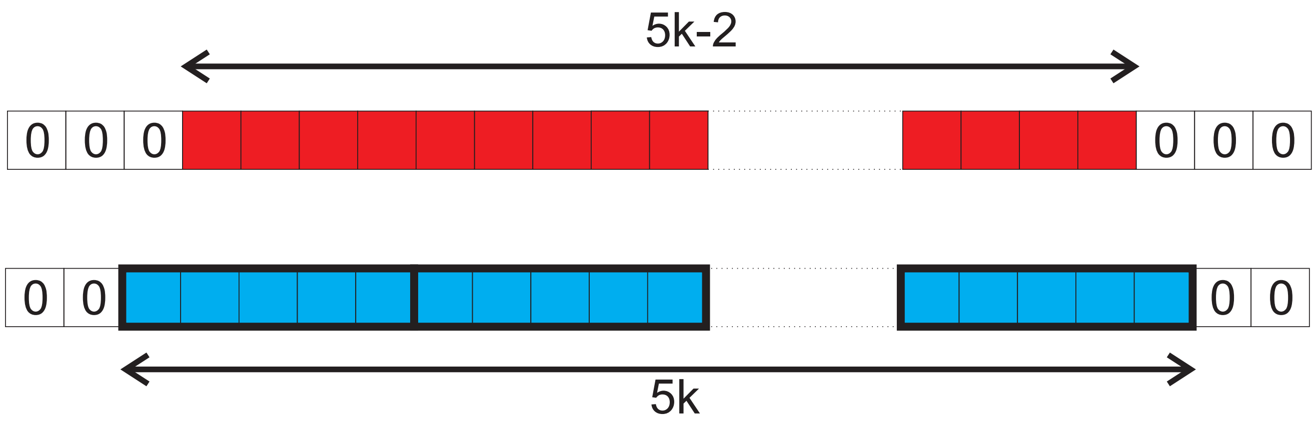
First the **proof idea** using rule 110.

As rule 110 is not surjective it has an orphan: 01010 of length five.

Let us see how this implies that there are different 0-finite configurations with the same image.

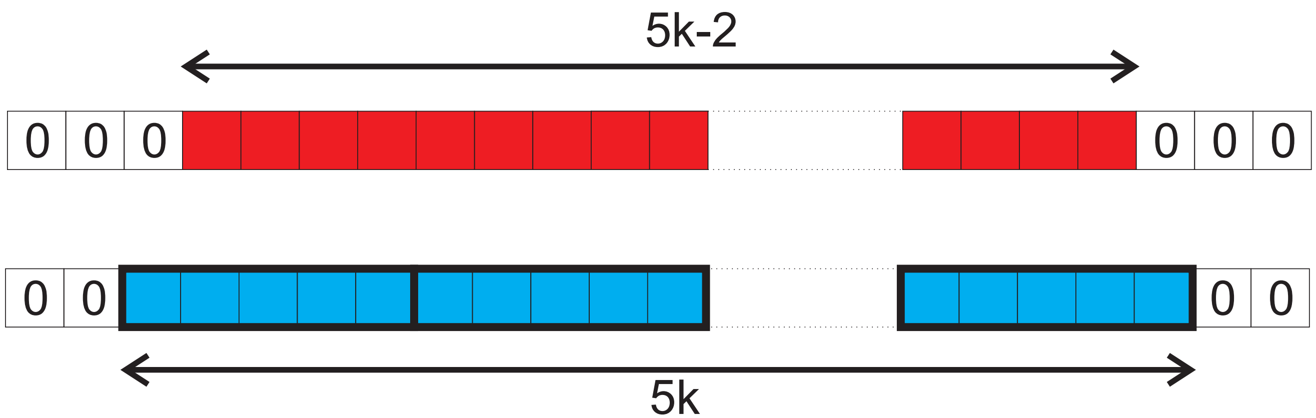


Consider a segment of length $5k-2$, for some k, and configurations c that are 0 outside of this segment. There are $2^{5k-2} = 32^k/4$ such configurations.



Consider a segment of length $5k-2$, for some k, and configurations c that are 0 outside of this segment. There are $2^{5k-2} = 32^k/4$ such configurations.

The non-0 part of $G(c)$ is within a segment of length $5k$. Partition this segment into k parts of length 5. Pattern 01010 cannot appear in any part, so only $2^5 - 1 = 31$ different patterns show up in the subsegments. There are at most 31^k possible configurations $G(c)$.



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As $32^k/4 > 31^k$ for large k, there are more choices for red than blue segments. So there must exist two different 0-finite configurations with the same image.

The same idea provides a general proof of the Myhill direction:

Proposition. If G is not surjective then G is not pre-injective.

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Corollary. If G_F is injective then G is surjective.

Proof. G_F injective $\implies G$ pre-injective $\implies G$ surjective

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Proof. G_F injective $\Longrightarrow$ G pre-injective $\Longrightarrow$ G surjective

Remark: We have now two different proofs for the implication

$$\mathbf{G} \text{ injective } \Longrightarrow \mathbf{G} \text{ surjective}$$

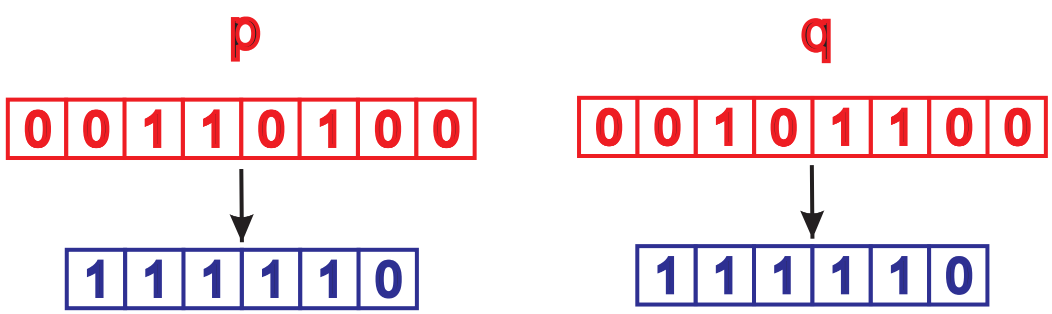
Namely

G injective $\Longrightarrow$ G pre-injective $\Longrightarrow$ G surjective
G injective $\Longrightarrow$ G_P injective $\Longrightarrow$ G_P surjective $\Longrightarrow$ G surjective

Moore direction: G not pre-injective $\Longrightarrow$ G not surjective.

First the **proof idea** using rule 110.

As rule 110 is not pre-injective it has two patterns with identical borders with the same image:

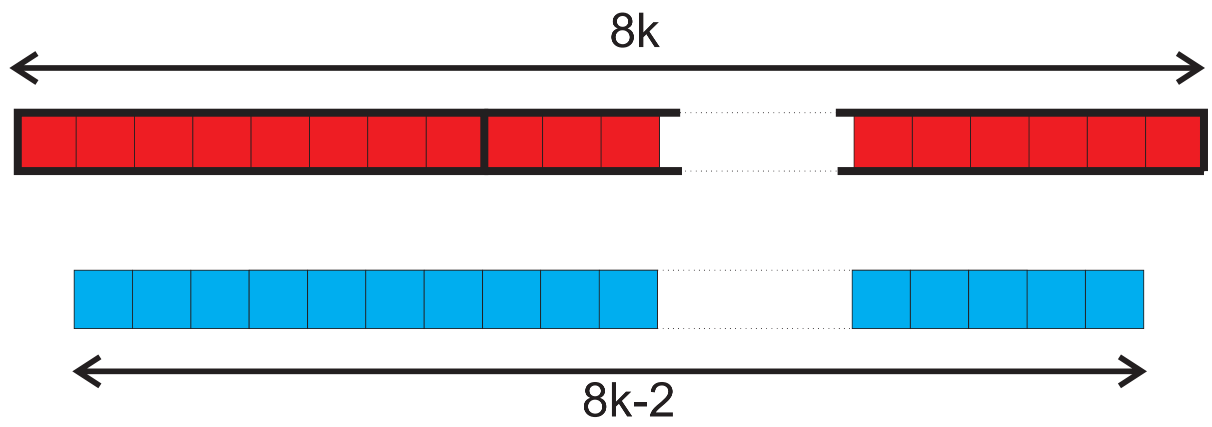


These patterns p and q of length 8 can be exchanged to each other in any configuration without affecting its image. There exist just

$$2^8 - 1 = 255$$

essentially different blocks of length 8.

Let us see how this implies that there exists an orphan.



Consider a segment of $8k$ cells, consisting of k parts of length 8. Patterns p and q are exchangeable, so the segment has at most 255^k different images.

There are, however, $2^{8k-2} = 256^k/4$ different patterns of size $8k-2$. Because $255^k < 256^k/4$ for large k, there are blue patterns without any pre-image.

The same idea provides a general proof of the Moore direction:

Proposition. If G is not pre-injective then G is not surjective.

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Proposition. If G is not pre-injective then G is not surjective.

Proof.

Corollary. If G is surjective then G_F is injective.

Proof. G surjective $\implies G$ pre-injective $\implies G_F$ injective

Examples

The **majority rule** is not surjective: finite configurations

$$\ldots 0000000 \ldots \quad \text{and} \quad \ldots 0001000 \ldots$$

have the same image, so G is not pre-injective.

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Pattern

$$01001$$

is an orphan.

Examples

In **Game-Of-Life** a lonely living cell dies immediately, so G is not pre-injective. GOL is hence not surjective.

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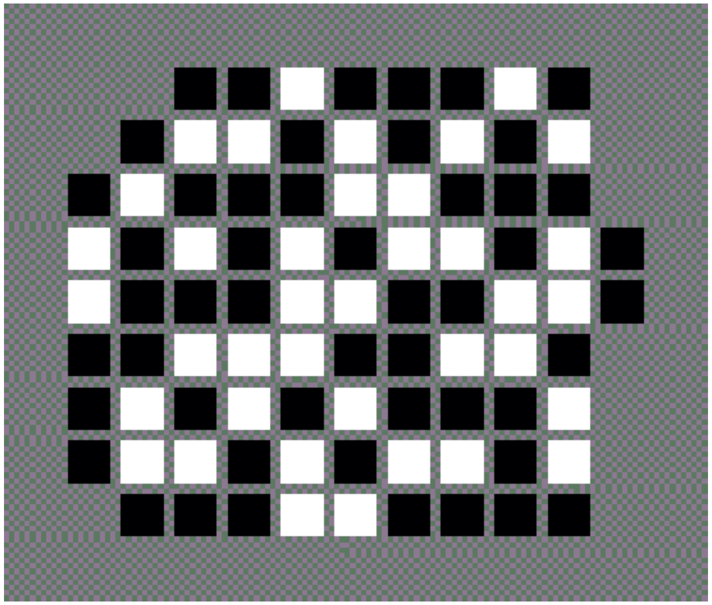
Interestingly, no small orphans are known for Game-Of-Life. Currently, the smallest known orphan consists of 88 cells (50 life, 38 dead):



Steven Eker (2017)

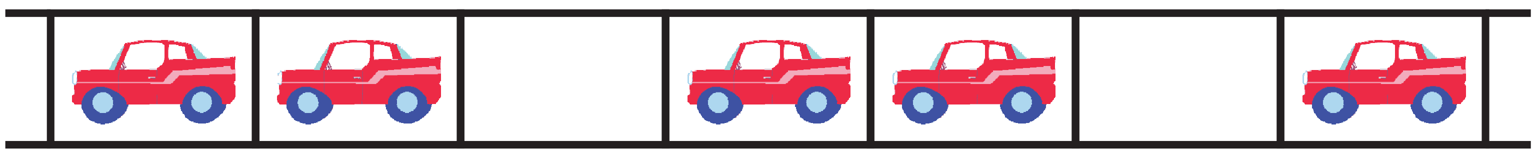
Examples

The **Traffic CA** is the elementary CA number 226.

$$\begin{array}{lcl} 111 & \longrightarrow & 1 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 0 \\ 010 & \longrightarrow & 0 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

The local rule replaces pattern 01 by pattern 10.

$$
\begin{aligned}
111 &\longrightarrow 1\\
110 &\longrightarrow 1\\
101 &\longrightarrow 1\\
100 &\longrightarrow 0\\
011 &\longrightarrow 0\\
010 &\longrightarrow 0\\
001 &\longrightarrow 1\\
000 &\longrightarrow 0
\end{aligned}
$$



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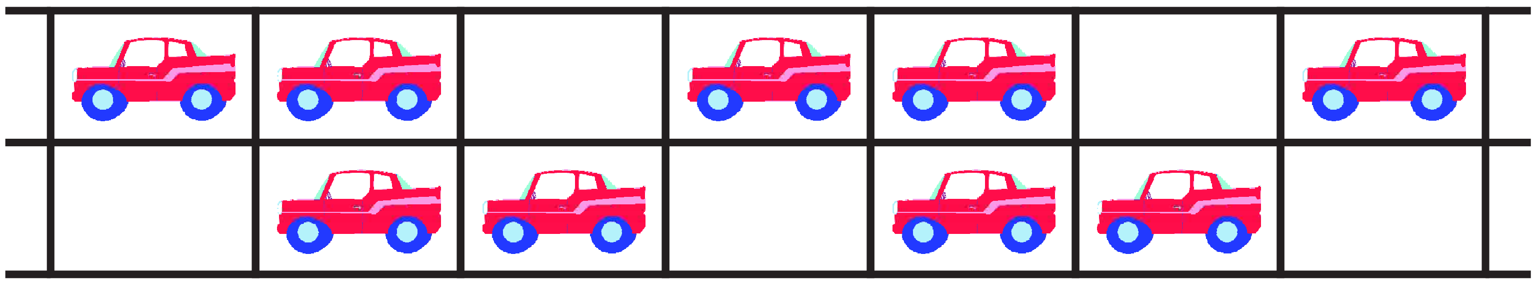

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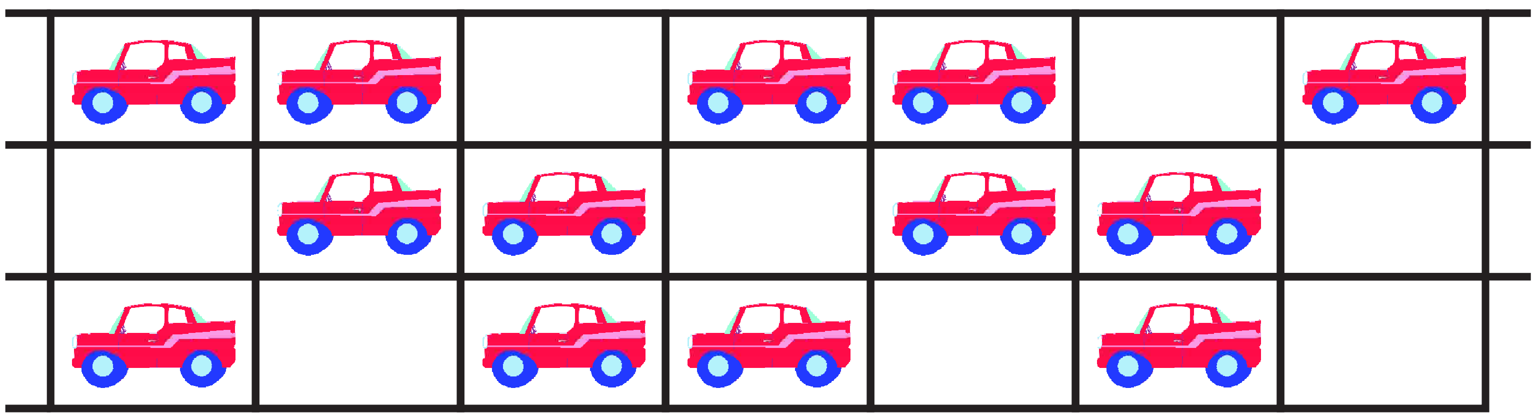

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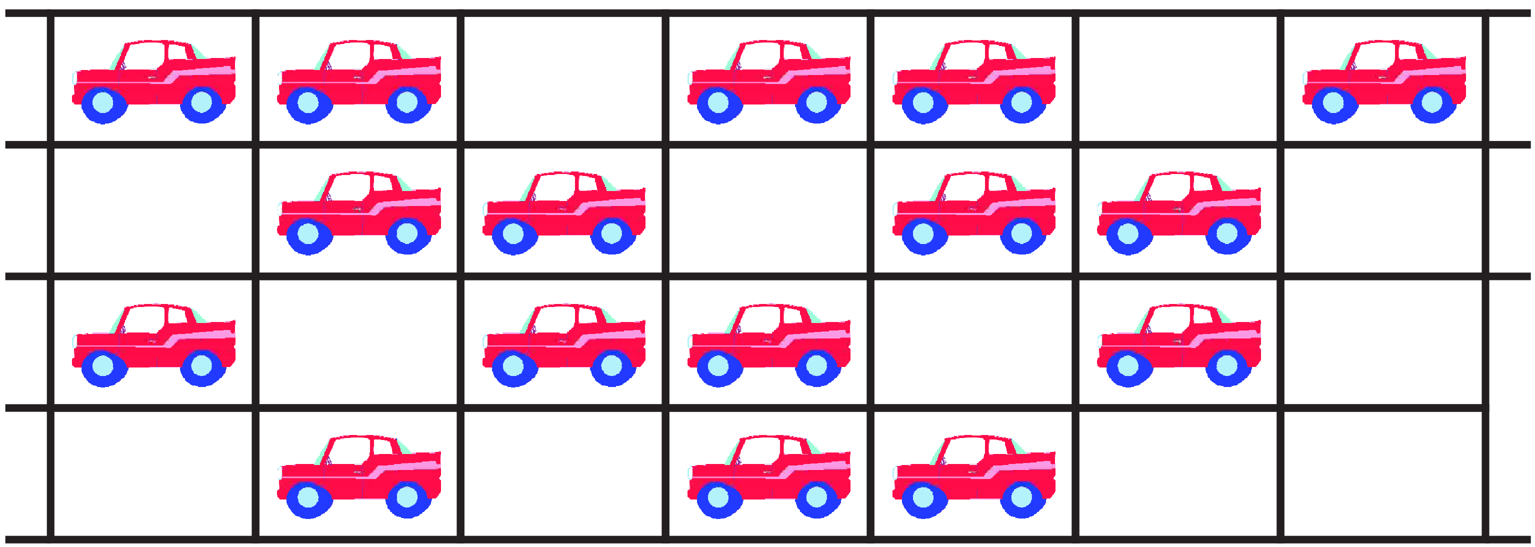
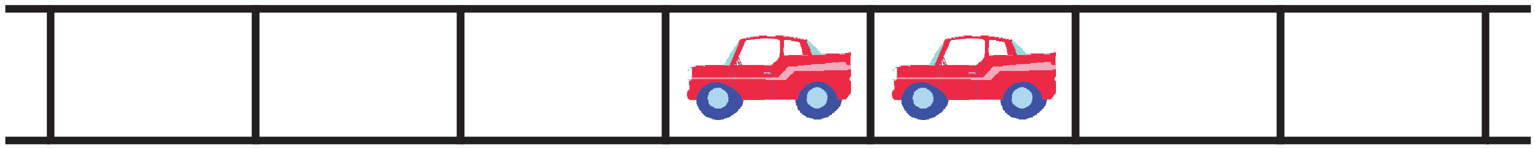
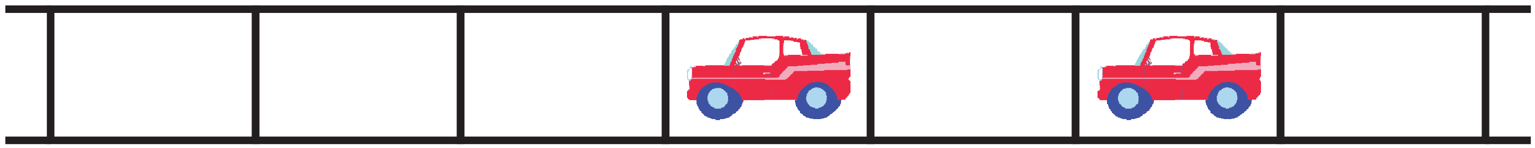


The local rule is balanced. However, there are two finite configurations with the same successor:





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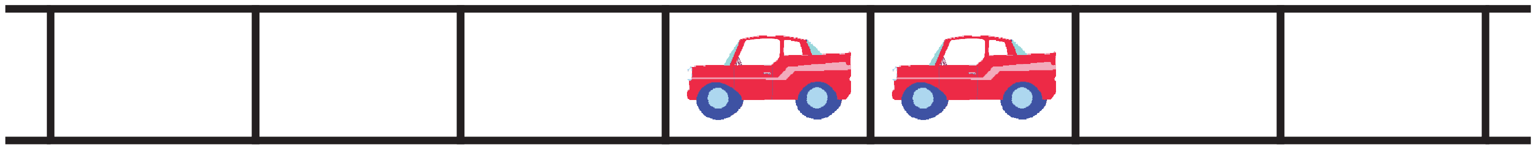

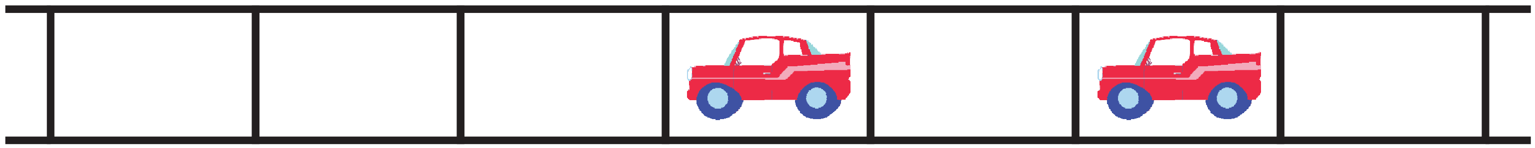

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There is an orphan of size four: