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The balance of surjective CA implies in the one-dimensional case the following:

Proposition. For every one-dimensional surjective CA there is a constant n such that every configuration has at most n pre-images.

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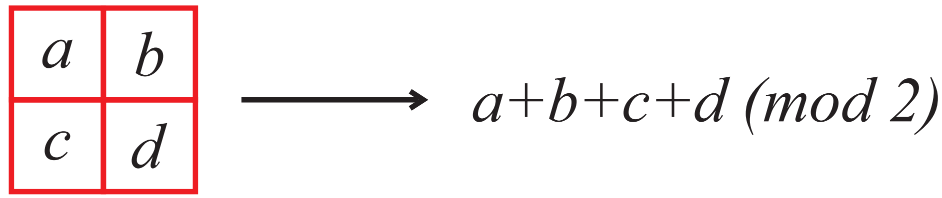
So among one-dimensional CA

$$\mathbf{G} \text{ surjective } \Longrightarrow \mathbf{G_P} \text{ surjective}$$

Example. The proposition and the corollary fail in 2D. Consider the **2D XOR automaton** with radius-$\frac{1}{2}$ neighborhood:



- This CA is **surjective**.
- The 0-uniform configuration c_0 has **uncountably many** pre-images, some of which are not spatially periodic: any modulo 2 sum of horizontal and vertical rows of 1's maps to c_0.
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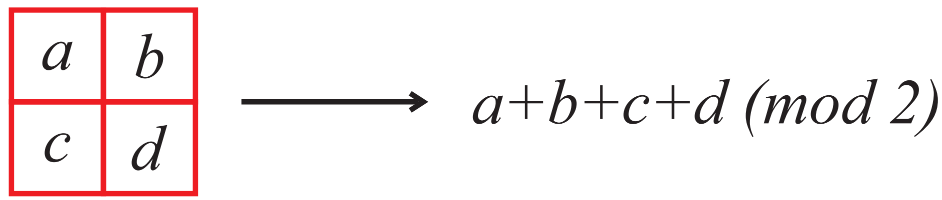

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It is not known whether in 2D

$$\mathbf{G} \text{ surjective } \stackrel{?}{\Longrightarrow} \mathbf{G_P} \text{ surjective}$$

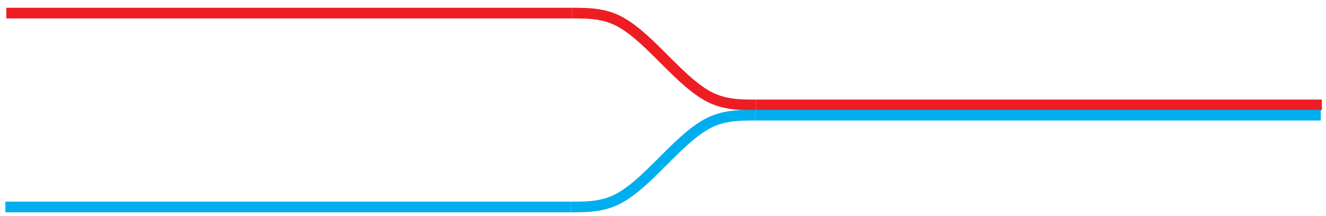
In non-surjective CA there is no finite bound on the number of pre-images. The proposition is valid in any dimension:

Proposition. If G is a **non-surjective** CA then there is a strongly periodic configuration with uncountably many pre-images.

Proof. Homework

One-dimensional configurations $c, e \in S^{\mathbb{Z}}$ are

- **positively asymptotic** if $\exists m \in \mathbb{Z}$ such that $c(i) = e(i)$ for all $i \geq m$,



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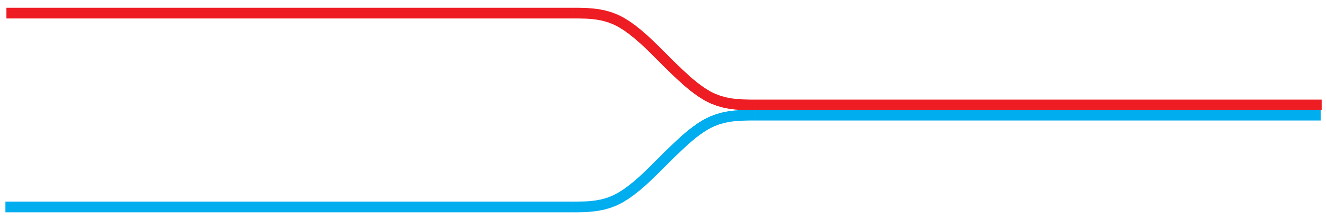
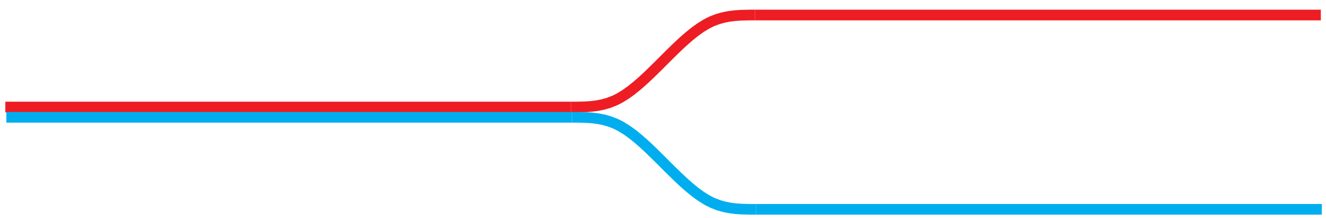


- **negatively asymptotic** if $\exists m \in \mathbb{Z}$ such that $c(i) = e(i)$ for all $i \leq m$,



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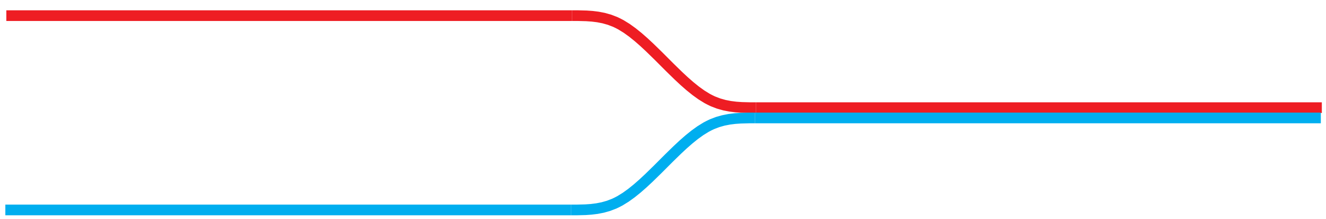

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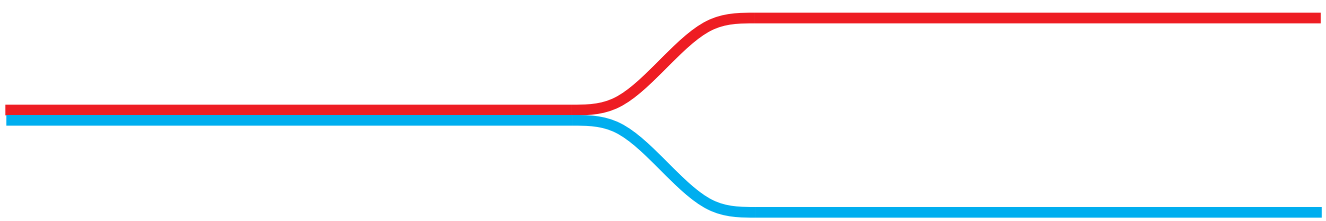
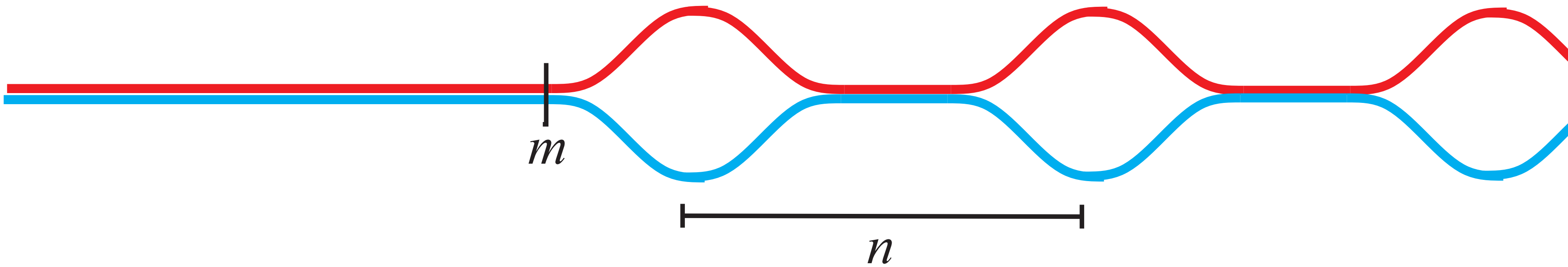


- **asymptotic** if it is both positively and negatively asymptotic.

One-dimensional configurations $c, e \in S^{\mathbb{Z}}$ are

- **positively n-separated** if $\exists m \in \mathbb{Z}$ such that for all $i \geq m$ holds:

$$c(i+j) \neq e(i+j) \text{ for some } j \in \{1, \ldots, n\}$$



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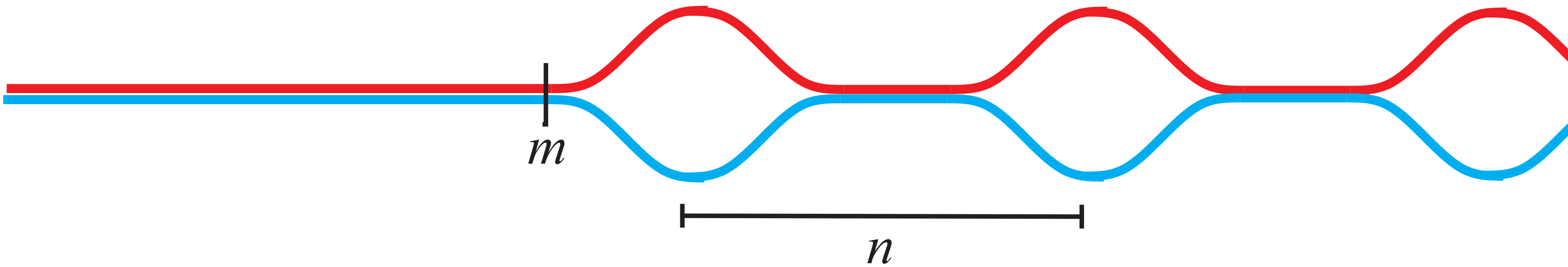

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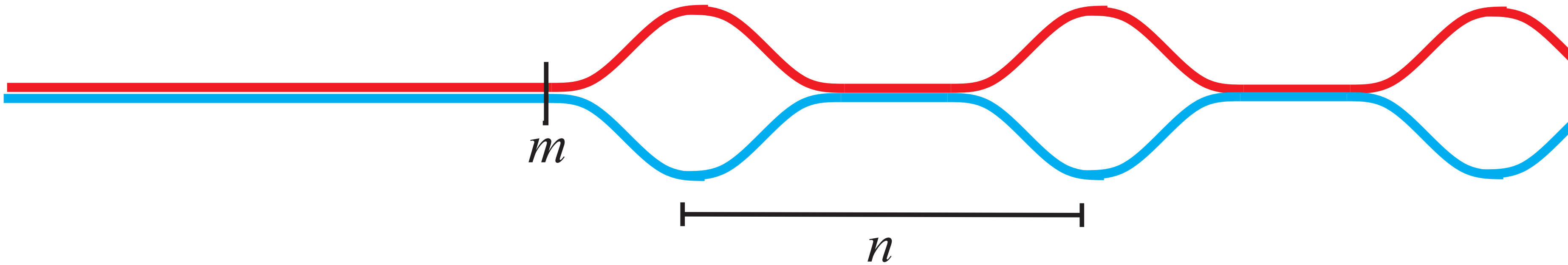

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- **totally n-separated** if for all $i \in \mathbb{Z}$ holds:

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One-dimensional configurations $c, e \in S^{\mathbb{Z}}$ are

- **positively separated** if they are positively n-separated for some n.
- **negatively separated** if they are negatively n-separated for some n.
- **totally separated** if they are totally n-separated for some n. (Equivalently: they are positively and negatively separated.)

Number m is a neighborhood **range** of a CA function G if G is defined by a CA whose neighborhood consists of m consecutive integers. (Some of which may be dummy neighbors.)

In particular, a radius-$\frac{1}{2}$ CA has neighborhood range 2, and a radius-r CA has range $2r + 1$, for any $r \in \mathbb{Z}_+$.

Proposition. Let G be a one-dimensional surjective CA function with neighborhood range m, and let $c, e \in S^{\mathbb{Z}}$ be such that $c \neq e$ and $G(c) = G(e)$. Then exactly one of the following three conditions is true:

(i) c and e are negatively asymptotic and positively $(m-1)$-separated,

(ii) c and e are positively asymptotic and negatively $(m-1)$-separated, or

(iii) c and e are both positively and negatively $(m-1)$-separated.

Proof.

Example. Three states $S = \{0, 1, 2\}$, radius-$\frac{1}{2}$ neighborhood, local rule

$$f(a,b) = \begin{cases} 2, & \text{if } a = 2, \\ 0, & \text{if } a \neq 2 \text{ and } a+b \text{ is even, and} \\ 1, & \text{if } a \neq 2 \text{ and } a+b \text{ is odd.} \end{cases}$$

(State 2 is unchanged. States 0 and 1 are changed using modulo 2 sum with the right neighbor.)

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The local rule is **left permutive**: For every $y, z \in S$ there is a unique x such that $f(x, y) = z$.

The CA is pre-injective and hence surjective: this follows from left-permutivity.

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- Configuration $\ldots 222 \ldots$ has a unique pre-image.
- Configuration $\ldots 000 \ldots$ has two pre-images.

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Configurations

$$\ldots 000020000 \ldots$$
$$\ldots 000021111 \ldots$$

are negatively asymptotic and positively 1-separated. They have the same image.

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$$\ldots 00000000 \ldots$$
$$\ldots 11111111 \ldots$$

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G is not injective but G_F is surjective if $q = 2$ is the quiescent state:

$$\mathbf{G_F} \text{ surjective } \not\Longrightarrow \mathbf{G} \text{ injective}$$

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G is surjective but G_F is not surjective if $q = 0$ is the quiescent state:

$$\mathbf{G} \text{ surjective } \not\Longrightarrow \mathbf{G_F} \text{ surjective}$$

Proposition. Among one-dimensional cellular automata, if G_P is injective then G is injective.

Proof.

We have proved all implications and non-implications in the figure of 1D CA: