Cellular Automata. Homework 3 (9.2.2026)

1. Determine if the following elementary CA are injective or surjective:
   (a) Rule 225.
   (b) Rule 142.
2. Prove: If CA function G is not surjective then there is a strongly periodic configuration $c \in \mathcal{P}$ that has infinitely many pre-images.
3. Let G be a one-dimensional CA. Prove that if a spatially periodic configuration has a pre-image then it has a spatially periodic pre-image.
4. Consider *Game-of-Life*.
   (a) Calculate how much imbalance there is in the local rule. More precisely, count the cardinalities of $f^{-1}(0) \subseteq \{0,1\}^9$ and $f^{-1}(1) \subseteq \{0,1\}^9$, where $f : \{0,1\}^9 \longrightarrow \{0,1\}$ is the local rule of Game-of-Life.
   (b) Using the result of (a) and the technique in the proof of Proposition 13, show that Game-of-Life has an orphan of size 40×40.
5. Consider *Game-of-Life*. Find two asymptotic configurations that have the same image and that differ only in one cell. Then, calculate an upper bound (probably a very large one!) on the size of an orphan for Game-of-Life using the technique in the proof of Proposition 18.
6. Let us call a one-dimensional configuration $c \in S^{\mathbb{Z}}$ *rich* (or shift transitive in $S^{\mathbb{Z}}$) if it contains a copy of every finite pattern over state set S. Let $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be a surjective CA function. Prove that c is rich if and only if $G(c)$ is rich. Does the analogous result hold also for d-dimensional CA when $d \geq 2$?
7. Toom's CA is the two-dimensional CA $(2, \{0,1\}, ((0,1),(0,0),(1,0)), f)$ where f is the majority function: $f(a,b,c) = 1$ if and only if $a+b+c \geq 2$. (Each cell takes the majority among its own state and the states in its upper and right neighbors.) Prove that every 0-finite configuration evolves into the 0-uniform configuration, and that every 1-finite configuration evolves into the 1-uniform configuration. (Recall: The s-uniform configuration has every cell in state s.)