Cellular Automata. Homework 4 (16.2.2026)

1. Let G be a one-dimensional surjective CA with neighborhood range m. Prove that if c is a spatially periodic configuration then all pre-images of c are totally $(m-1)$-separated.

2. Find a one-dimensional surjective CA that has three configurations c_1, c_2, c_3 such that $G(c_1) = G(c_2) = G(c_3)$ and

   - c_1 and c_2 are negatively asymptotic and positively separated,
   - c_1 and c_3 are positively asymptotic and negatively separated, and
   - c_2 and c_3 are positively and negatively separated.

   (This shows that the three possibilities in Proposition 23 can all occur in the same CA, and even among the pre-images of the same configuration.)

3. Let us call a one-dimensional configuration $c \in S^{\mathbb{Z}}$ (translationally) *recurrent* if every finite subpattern of c appears infinitely many times in c. Let us call c (translationally) *uniformly recurrent* if for every finite subpattern p of c there exists a positive integer n such that every segment of length n in c contains a copy of p.

   Let $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be a CA function. Prove:

   (a) If c is recurrent then $G(c)$ is recurrent.

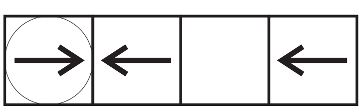
   (b) If c is uniformly recurrent then $G(c)$ is uniformly recurrent.

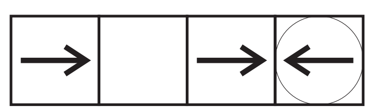
4. Prove that if G is a surjective one-dimensional CA and $c \in S^{\mathbb{Z}}$ is uniformly recurrent (see the previous problem), then c has a uniformly recurrent pre-image. (Hint: Consider a pre-image e of c that has a minimal set of finite subpatterns in the sense that there is no other pre-image of c whose finite subpatterns would be properly included in the set of the subpatterns of e. Why does such an e exist and why is e uniformly recurrent ?)

5. Construct the labeled de Bruijn -graph of elementary CA 218. Find the shortest orphans of this CA by using the subset construction.

6. Describe (i.e. give an algorithm) how one can use the de Bruijn -representation of a one-dimensional CA to determine if the CA is surjective on q-finite configurations, for a given quiescent state q. Apply your method on the elementary CA 216 to determine a shortest word $w \in \{0,1\}^*$ such that the configuration $\ldots 000\ w\ 000 \ldots$ does not have a 0-finite pre-image.

7. Consider the following one-dimensional CA (due to Gacs, Kurdiumov, Levin): The state set is $S = \{\leftarrow, \rightarrow\}$. A cell changes the direction of its arrow if and only if there are opposing arrows at the first and the third neighbor on the side pointed by the arrow. In other words, the circled arrow is swapped in the following contexts:





   Prove that every $\rightarrow$-finite configuration becomes eventually $\rightarrow$-uniform, and prove that every $\leftarrow$-finite configuration becomes eventually $\leftarrow$-uniform. (Cf. the two-dimensional Toom's rule in the previous homework set.)

   (Hint: Let c be $\rightarrow$-finite. Consider the finite patterns $p_1 = \leftarrow\ \leftarrow$ and $p_2 = \leftarrow\ \rightarrow\ \rightarrow\ \leftarrow$. Prove that the position of the leftmost symbol of the leftmost occurrence of pattern p_1 or p_2 in c moves at least one cell to the right each time step, until there is no occurrence of p_1 and p_2. Then show that once there is no p_1 and p_2, the position of the rightmost symbol $\leftarrow$ moves at least three positions to the left, until all cells are in state $\rightarrow$.)