Cellular Automata. Homework 5 (23.2.2026)

1. Construct the pair graph of elementary CA 218. Using this graph verify that the CA is not surjective, and find a pair of 0-finite configurations c and e such that $G(c) = G(e)$ while $c \neq e$.

The next three problems involve the *traffic CA*, the elementary CA where state 0 becomes 1 if and only if its right neighbor was in state 1, and state 1 becomes 0 if and only if its left neighbor was in state 0. In other words, each occurrence of pattern 01 gets replaced by pattern 10. (Intuition: each 1 represents a left moving car. The car can move to the left if and only if there is an empty slot on its left. The car can not move if the left neighbor is already occupied by another car.)

2. Determine the Wolfram number of the traffic CA. Using the de Bruijn -graph determine if the traffic CA is injective or surjective.

3. The traffic CA has the obvious property that, for every 0-finite configuration c, the number of cells in state 1 in c and in $G(c)$ are the same. We say that the number of 1's is *conserved*. Prove that the number of occurrences of pattern 01 in every 0-finite configuration c is *non-decreasing*, that is, the number of occurrences of 01 in $G(c)$ is at least the same as the number of its occurrences in c.

4. Consider the following one-dimensional CA over the state set $\{0, 1\}$: The local rule consists of the following two steps:

   (1) Apply the traffic CA,

   (2) Change the 1 into 0 in every pattern 0100, and change the 0 into 1 in every pattern 1101.

   Prove that this CA has the same property as the CA by Gacs, Kurdiumov and Levin, studied in problem 7, homework set 4: Every 0-finite configuration becomes eventually 0-uniform, and every 1-finite configuration becomes eventually 1-uniform.

5. Let q be a quiescent state. Let G be a CA function such that for every configurations c, there is number $t \geq 0$ such that $G^t(c)$ is the q-uniform configuration. Such CA are called *nilpotent*. Prove that if G is nilpotent then there is number t (independent of c) such that for all configurations c, the configuration $G^t(c)$ is q-uniform. (Hint: Consider a rich configuration.)

6. Let us call a CA G *periodic* if G^n is the identity function, for some $n \geq 1$, and let us call it *eventually periodic* if for some n and $p > 0$ holds $G^n = G^{n+p}$. Prove the following:

   (a) If all configurations are (temporally) periodic then the CA is periodic.

   (b) If all configurations are (temporally) eventually periodic then the CA is eventually periodic.

7. Amman's Wang tile set (Figure 23 in the notes) is aperiodic. Prove that there exists a reversible one-dimensional cellular automaton with 36 states and neighborhood $(0, 1)$ (=radius-$\frac{1}{2}$), and a subset A of the state set such that

   - there is an orbit in which every cell is always in a state that belongs to A (that is, $F^t(c)(i) \in A$ for all $t, i \in \mathbb{Z}$), but
   - there is no such periodic orbit, that is, if $F^n(c) = c$ for some $n \geq 1$ then there are $t, i \in \mathbb{Z}$ such that $F^t(c)(i) \notin A$.