Cellular Automata. Homework 6 (2.3.2026)

In the undecidability questions you may use the known undecidability results concerning Wang tiles given on the reverse side of this paper.

1. The following decision problems are undecidable. Determine if they are semi-decidable, and if their complement problem is semi-decidable.
   (a) NILPOTENCY: Is a given cellular automaton nilpotent ? (See Problem 5 from last week for the definition of nilpotency.)
   (b) PERIODICITY: Is a given cellular automaton periodic ? (See Problem 6 from last week for the definition of periodicity.)
2. Assume that we know that it is undecidable if a given 2D CA is reversible. Show how this implies that there are non-injective cellular automata that are injective on strongly periodic configurations.
3. Prove that it is undecidable if a given two-dimensional CA has any fixed point configurations. Is the question decidable or undecidable for one-dimensional CA ?
4. Let A be a fixed CA with quiescent state q. The GARDEN-OF-EDEN PROBLEM for A asks whether a given q-finite configuration is a Garden-of-Eden. Prove that there exists a two-dimensional CA whose GARDEN-OF-EDEN PROBLEM is undecidable. Is the question decidable or undecidable for one-dimensional CA ?
5. A CA G with quiescent state q is called *nilpotent on periodic configurations* if every strongly periodic configuration evolves into the quiescent configuration.
   (a) Prove that every nilpotent CA is also nilpotent on periodic configurations, but the converse is not true.
   (b) Prove that it is undecidable if a given two-dimensional CA is nilpotent on periodic configurations.
6. A CA G with quiescent state q is called *nilpotent on finite configurations* if every finite configuration evolves into the quiescent configuration, that is, for every finite configuration c there is positive integer n such that all cells in $G^n(c)$ are quiescent.
   (a) Prove that every nilpotent CA is also nilpotent on finite configurations, but the converse is not true.
   (b) Prove that it is undecidable if a given two-dimensional CA is nilpotent on finite configurations.
7. Consider the two-dimensional CA constructed in the proof of Proposition 46 for a given tile set T.
   (a) If the tile set T admits a tiling of the plane, is the corresponding CA periodic ? (Recall: CA G is periodic if G^n is the identity function for some $n \geq 1$.)
   (b) If the tile set T does not admit a tiling of the plane, is the corresponding CA periodic ?
   (c) Is it decidable to determine if a given two-dimensional CA is periodic ?

The following decision problems have been proved undecidable in the *Tilings and Patterns* class. Here we use them without a proof:

TILING PROBLEM
Instance: A finite set T of Wang tiles
Problem: Does T admit a valid tiling ?

TILING PROBLEM WITH THE SEED TILE
Instance: A finite set T of Wang tiles and a seed tile $s \in T$
Problem: Does there exist a valid tiling t of the plane such that $t(0,0) = s$?

FINITE TILING PROBLEM
Instance: A finite set T of Wang tiles and a blank tile $b \in T$
Problem: Does there exist a valid tiling t such that $\{(i,j) \in \mathbb{Z}^2 \mid t(i,j) \neq b\}$ is finite but non-empty?

PERIODIC TILING PROBLEM
Instance: A finite set T of Wang tiles
Problem: Does T admit a periodic tiling ?

There is a fixed set T of Wang tiles such that the following decision problem is undecidable:

COMPLETION PROBLEM FOR TILE SET T
Instance: A finite pattern p
Problem: Does T admit a valid tiling that contains a copy of p ?