Undecidable problems concerning Wang tiles

These decision problems were proved undecidable in "Tilings and Patterns":

Tiling problem

Instance: A finite set T of Wang tiles

Positive instance: T admits a valid tiling.

Periodic tiling problem

Instance: A finite set T of Wang tiles

Positive instance: T admits a periodic valid tiling.

Finite tiling problem

Instance: A finite set T of Wang tiles and a blank tile $b \in T$

Positive instance: T admits a valid tiling that is not b-finite but not b-uniform.

Tiling problem with a seed tile

Instance: A finite set T of Wang tiles and a seed tile $s \in T$

Positive instance: T admits a valid tiling that contains s.

Proofs of undecidability use reductions from TM halting on blank tape.

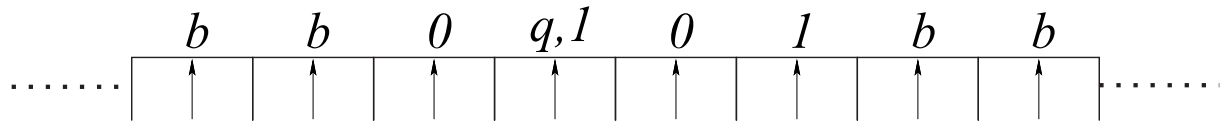
We associate to each Turing machine M a Wang tile set $\mathcal{P}_M$ whose tilings simulate iterations of M.

Horizontal rows of tilings represent configurations on consecutive time instances, time increasing upwards.

Colors will be represented as labeled arrows: In neighboring tiles arrow heads and tails with identical labels must meet each other.

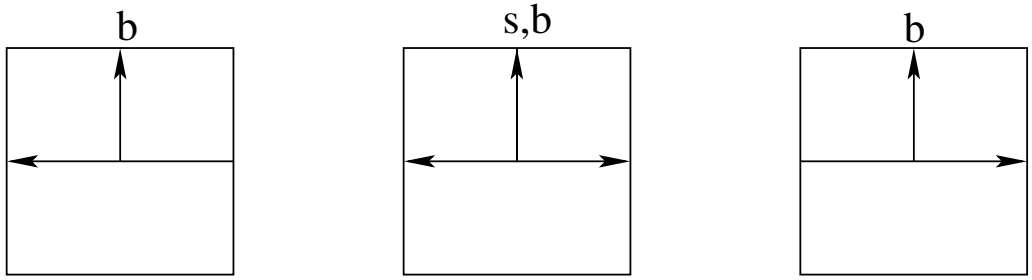
The labels are

- tape symbols (representing a tape location containing that symbol), or
- state/tape symbol pairs (representing a tape location containing the control unit at the given state).



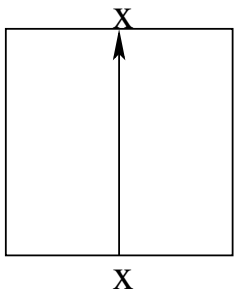
Let $M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$. In $\mathcal{P}_M$ we have the following tiles:

(i) three starting tiles to represent the blank tape



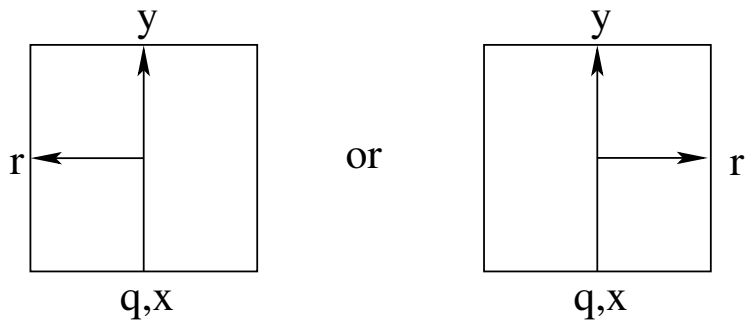
Let $M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$. In $\mathcal{P}_M$ we have the following tiles:

(ii) for every tape letter $x \in \Gamma$ an alphabet tile



Let $M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$. In $\mathcal{P}_M$ we have the following tiles:

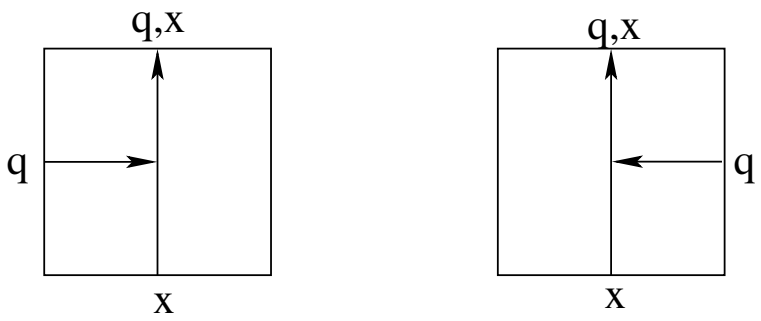
(iii) for every non-halting state $q \in S \setminus \{h_a, h_r\}$ and tape symbol $x \in \Gamma$ one action tile



where the left tile is used iff $\delta(q, x) = (r, y, -1)$ and the right tile iff $\delta(q, x) = (r, y, +1)$,

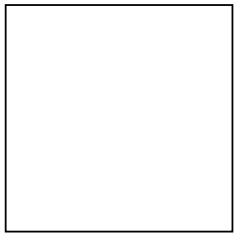
Let $M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$. In $\mathcal{P}_M$ we have the following tiles:

(iv) for every non-halting state $q \in S \setminus \{h_a, h_r\}$ and tape symbol $x \in \Gamma$ the two merging tiles



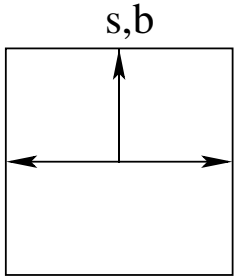
Let $M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$. In $\mathcal{P}_M$ we have the following tiles:

(v) the blank tile



Example. Many-one reduction from TM halting on blank tape to Tiling problem with a seed tile:

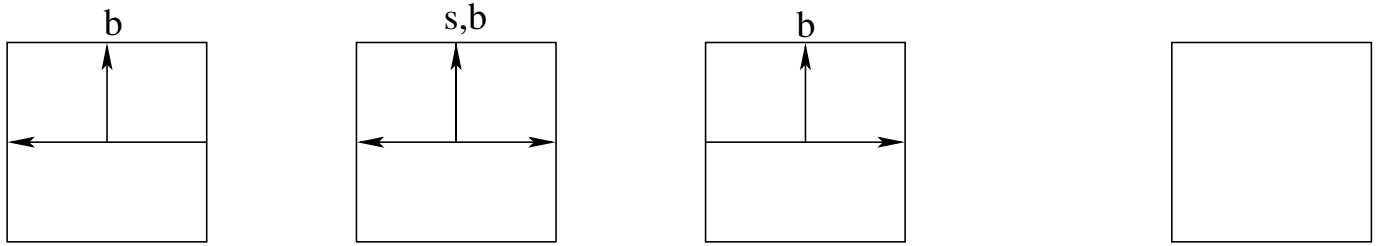
For a given TM M construct the tile set $\mathcal{P}_M$ and set



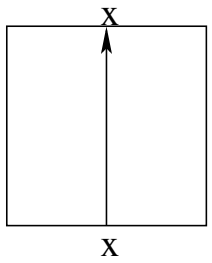
as the seed tile.

Tile set $\mathcal{P}_M$ can be effectively constructed. It is now enough to note that $\mathcal{P}_M$ admits a tiling containing the seed tile if and only if M does not halt from the empty initial tape.

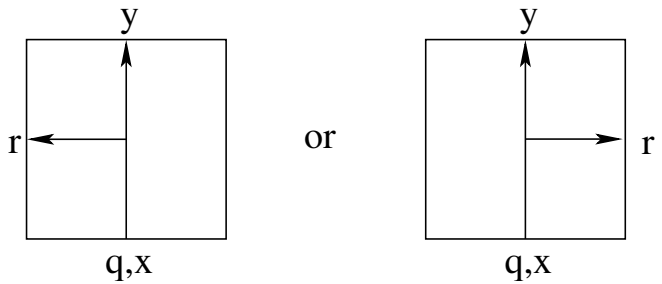
$$M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$$



$\forall x \in \Gamma$:



$\forall x \in \Gamma, \forall q \in S \setminus \{h_a, h_r\}$:



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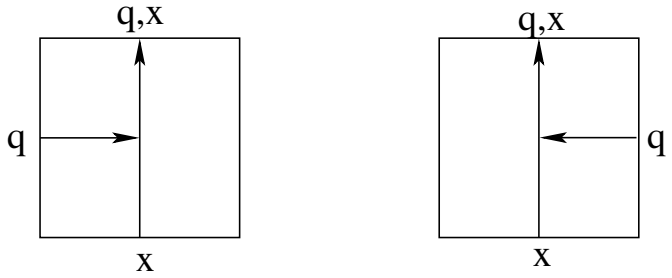


Note: Tiling problem with a seed tile is not the same problem as Tiling problem.

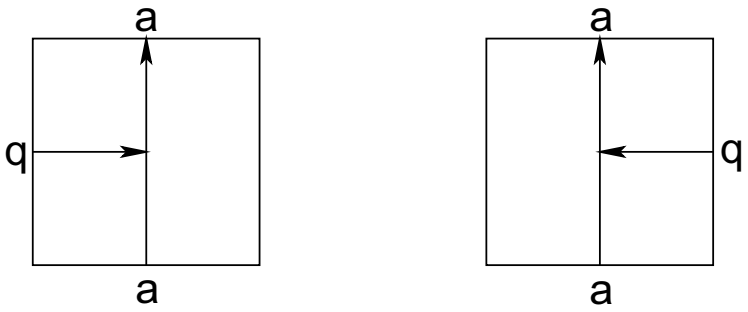
The request for the named seed tile to appear in the tiling makes the proof easy, as it allows to guarantee the proper initialization of the Turing machine computation.

Undecidability of Tiling problem is significantly harder to prove. (Is done in "Tilings and Patterns".)

Example. Many-one reduction from TM halting on blank tape to Finite tiling problem.

For a given TM M construct the tile set $\mathcal{P}_M$ and add to it the following tiles:

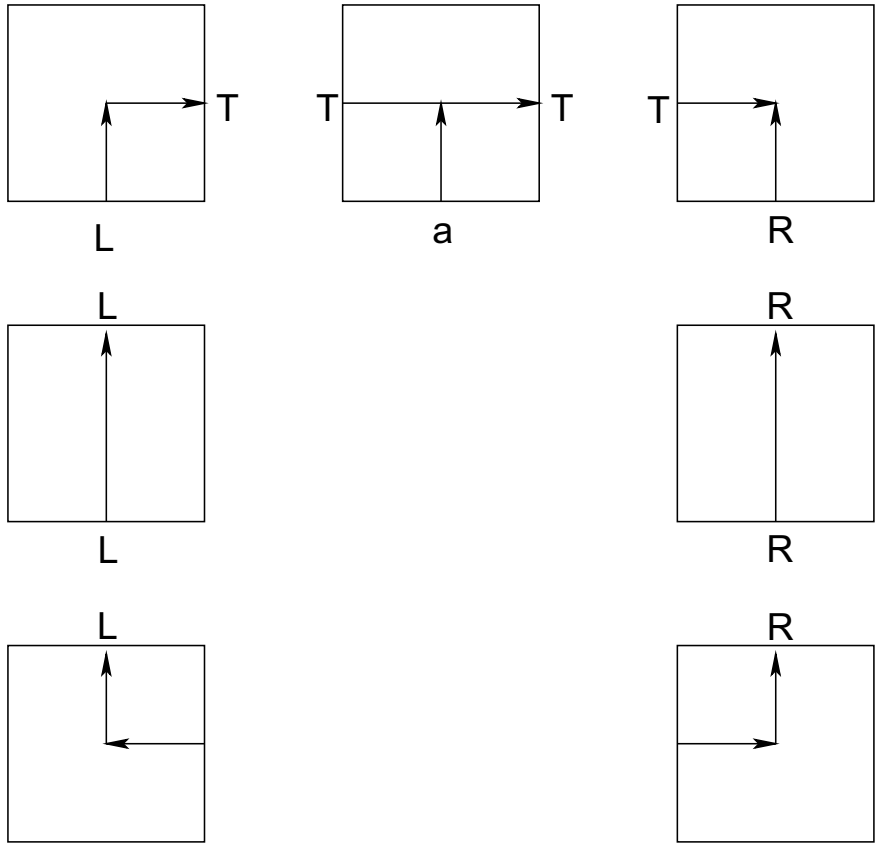
• For both halting states $q \in \{h_a, h_r\}$ and all tape symbols $a \in \Gamma$ the two killer tiles



Example. Many-one reduction from TM halting on blank tape to Finite tiling problem.

For a given TM M construct the tile set $\mathcal{P}_M$ and add to it the following tiles:

- The boundary tiles where R, L, T are new symbols and $a \in \Gamma$



Let B be the blank tile. The tile set we constructed admits a B-finite tiling that is not B-uniform if and only if M halts from the empty initial tape.

It follows that Finite tiling problem is undecidable.

Let T be a fixed Wang tile set. The following decision problem asks whether a given finite pattern of tiles appears in some valid tiling by T:

Completion Problem for Tile Set T

Instance: A finite pattern p

Positive instance: T admits a tiling that contains a copy of p.

Here T is not part of the input but rather a fixed parameter. The problem is different for different choices of T. For some tile sets the completion problem is decidable, but (as proved below) there are also tile sets T whose completion problem is undecidable.

The situation is analogous to a decision problem for a fixed Turing machine M:

Halting problem of TM M

Instance: An input word $w \in \Sigma^*$

Positive instance: M halts on input w.

Also here, there are Turing machines with decidable halting problem, but also machines with undecidable halting problem:

Proposition. There exists a Turing machine M whose Halting problem of TM M is undecidable.

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$\Longrightarrow$ There exists a recursively enumerable language L that is not recursive.
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$\Longrightarrow$ There exists a Turing machine M whose Halting problem of TM M is undecidable
(For example M that recognizes L, looping if a word is not accepted) $\square$

Now we can many-one reduce the problem

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Instance: An input word $w \in \Sigma^*$

Positive instance: M halts on input w.

of any TM M to the problem

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for a suitable tile set T:

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Instance: A finite pattern p
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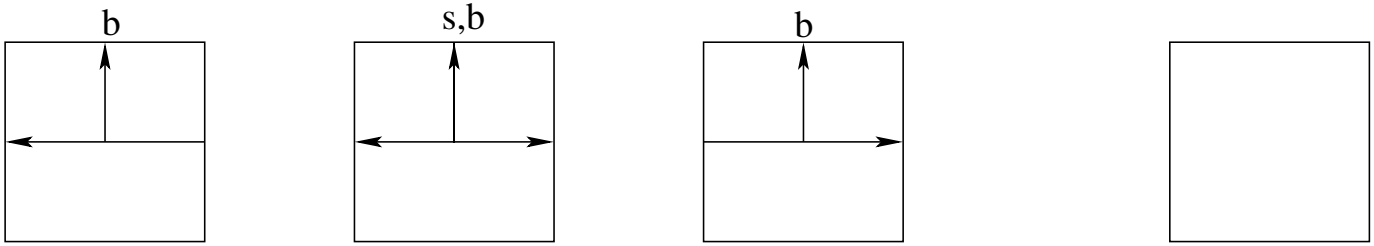
for a suitable tile set T:

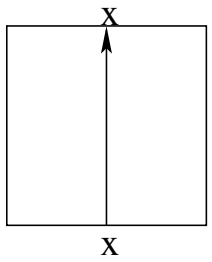
Proposition. There exists a Wang tile set T such that Completion Problem for Tile Set T is undecidable.

Proof. Let M have undecidable Halting problem of TM M. Let $\mathcal{P}_M$ be the tile set associated to M.

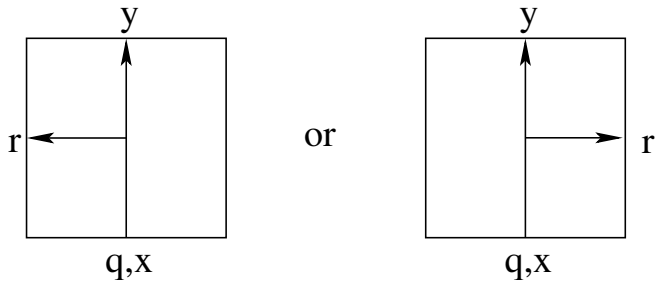
$$M = (S, \Gamma, \Sigma, \delta, s, h_a, h_r, b)$$



$\forall x \in \Gamma$:



$\forall x \in \Gamma, \forall q \in S \setminus \{h_a, h_r\}$:



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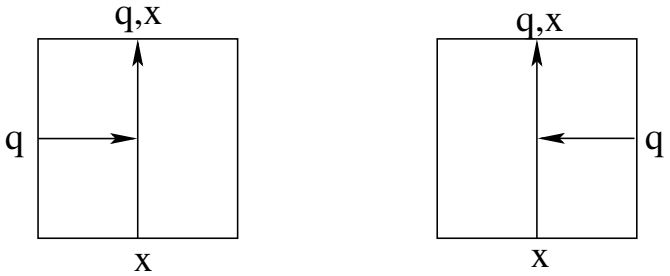
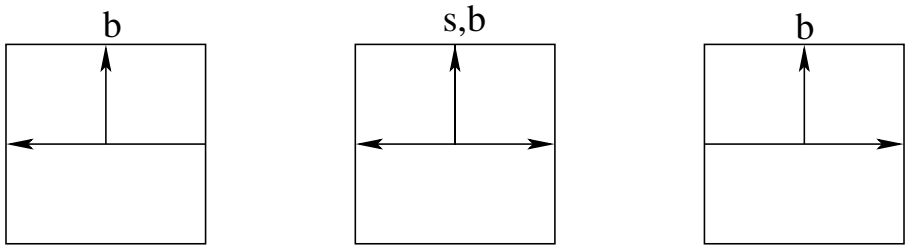
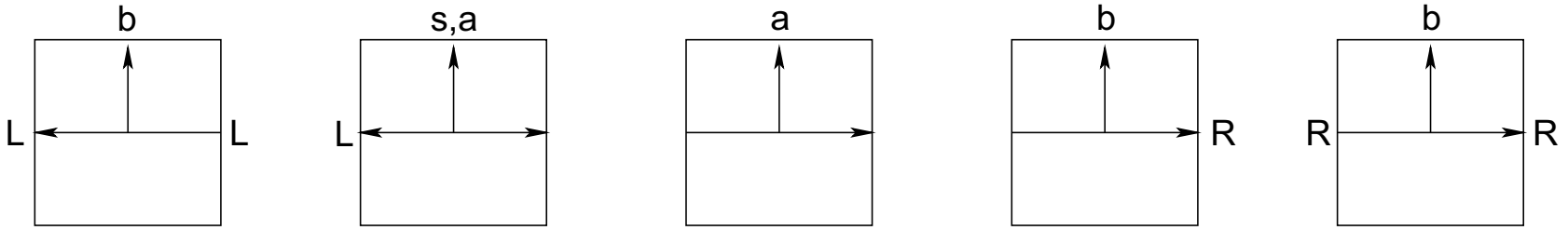


We modify the tile set a little, in order to give input words as finite patterns: We replace the three starting tiles

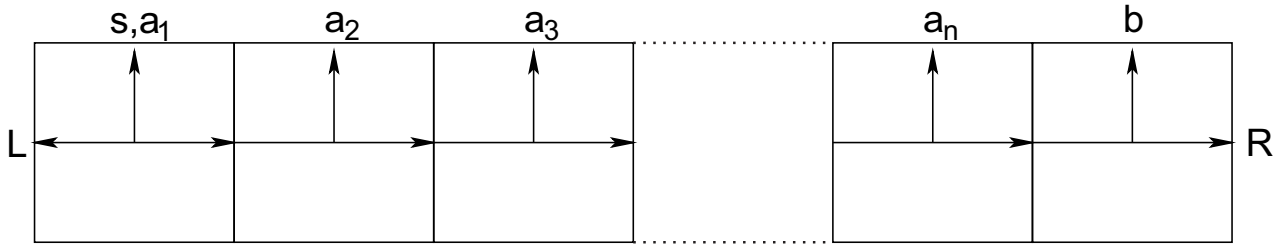


by tiles



for all $a \in \Sigma$. Here b is the blank tape letter and L and R are new labels.

It is now enough to show that M halts on input word $w = a_1 a_2 \dots a_n$ if and only if the tiles no not admit a tiling that contains the pattern



Restricting the instances of Tiling problem to NW-deterministic Wang tile sets keeps the problem undecidable.

NW-deterministic tiling problem

Instance: A finite NW-deterministic set T of Wang tiles

Positive instance: T admits a valid tiling.

Proposition. The problem NW-deterministic tiling problem is undecidable.

Proof. Skipped.

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Instance: A finite NW-deterministic set T of Wang tiles

Positive instance: T admits a valid tiling.

Proposition. The problem NW-deterministic tiling problem is undecidable.

In fact: Even restricting the instances to Wang tile sets that are simultaneously NW-, NE-, SW- and SE-deterministic keeps the tiling problem undecidable!