

Undecidable problems concerning CA

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2D nilpotency

Instance: A two-dimensional cellular automaton G

Positive instance: G is nilpotent.

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Proposition. Decision problem **2D nilpotency** is undecidable. It is semi-decidable while the complement is not semi-decidable.

Proof. A many-one reduction from **Tiling problem**.

The reduction: For a given Wang tile set T , construct the following 2D CA.

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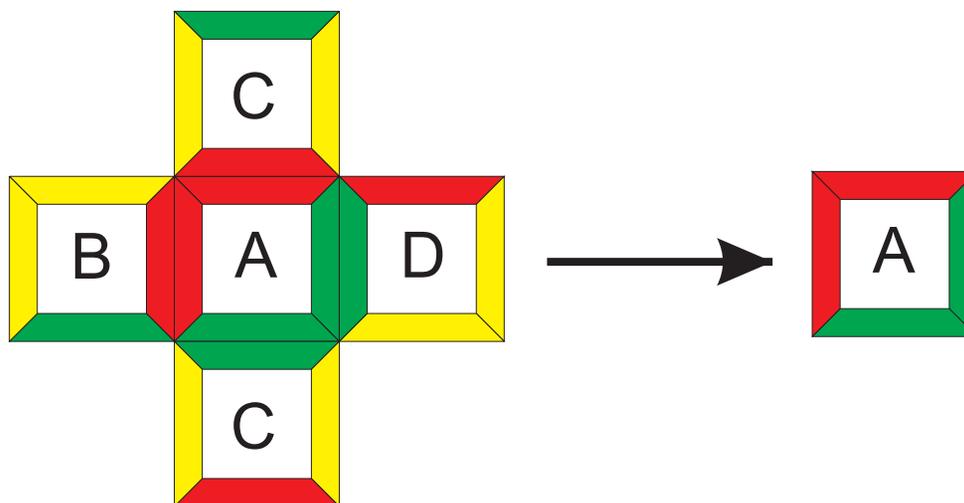
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- The local rule keeps state unchanged if all states in the neighborhood are tiles with matching colors. In all other cases the new state is q .

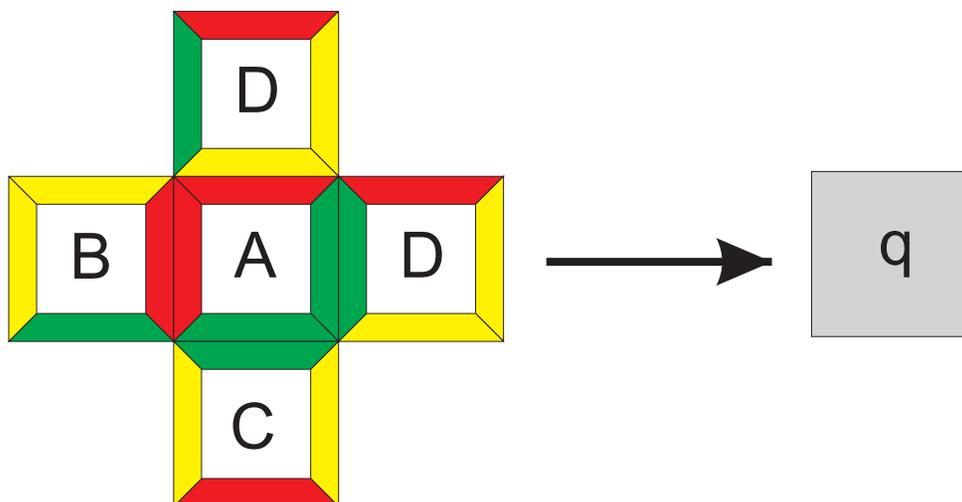
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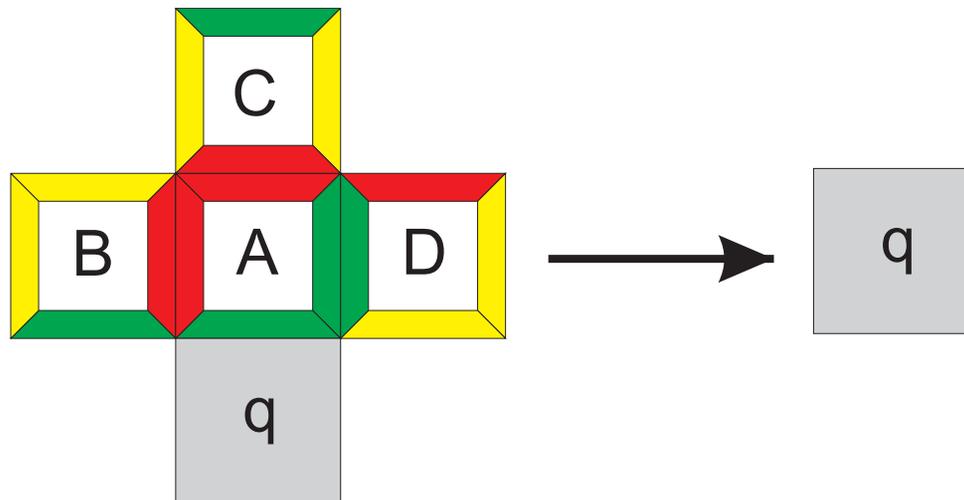
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Properties:

(1) Valid tilings are fixed points. So if T admits a valid tiling the CA is not nilpotent.

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Properties:

(1) Valid tilings are fixed points. So if T admits a valid tiling the CA is not nilpotent.

(2) If T does not admit a valid tiling then, for some n , there is no tiling of the $n \times n$ square. For any initial configuration c then $G(c)$ contains state q in every $n \times n$ square. State q spreads so eventually $G^t(c)$ becomes q -uniform. The CA is nilpotent.

1D nilpotency

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Proof. A many-one reduction from **NW-deterministic tiling problem**.

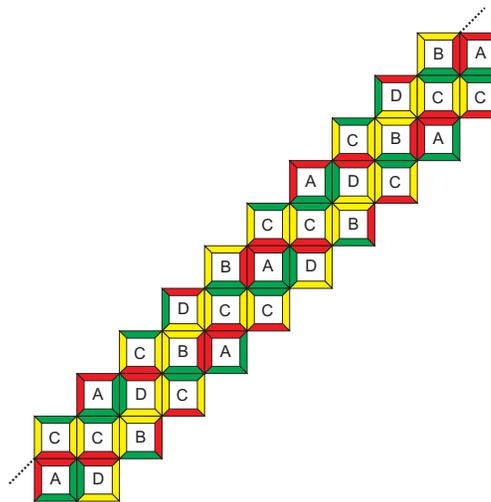
For a given NW-deterministic Wang tile set T , construct the following 1D CA.

- state set is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,
- neighborhood is $(0, 1)$,
- local rule $f : S^2 \longrightarrow S$ is defined as follows:

– $f(A, B) = C$ if the colors match in



– $f(A, B) = q$ if $A = q$ or $B = q$ or no matching tile C exists.



If configurations are drawn along diagonals, rule f puts matching tiles on the diagonal below.

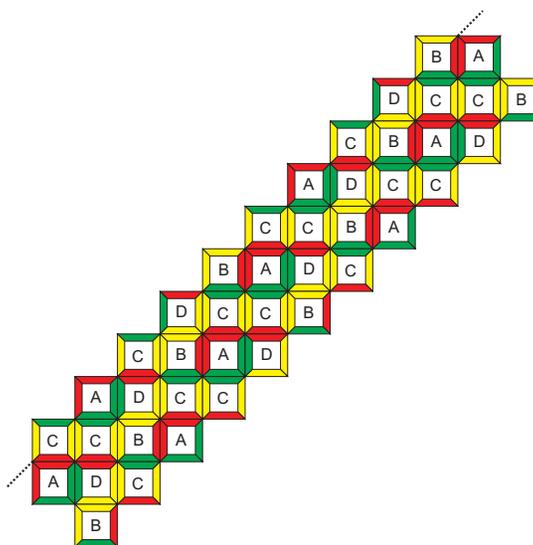
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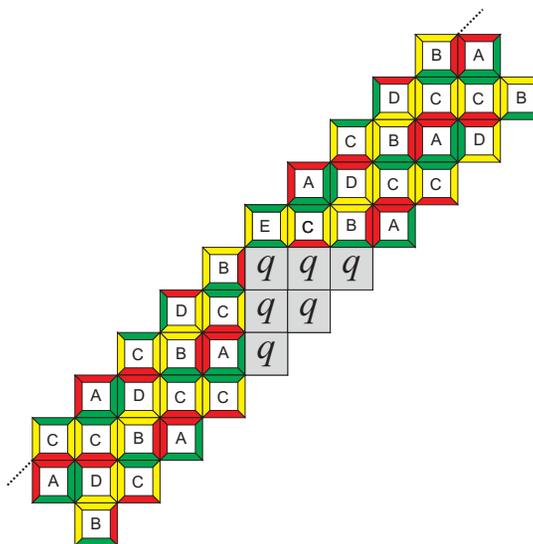
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If no matching tile exists, a spreading “killer” state q is created.

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(2) If T does not admit a valid tiling then, for some n , there is no tiling of the $n \times n$ square. For any initial configuration c , if some cell i does not have state q in $G^{2n}(c)$ then it is the lower right corner of a correctly tiled $n \times n$ square in the space-time diagram. So $G^{2n}(c)$ must be q -uniform. The CA is nilpotent.

2D reversibility

Instance: A two-dimensional cellular automaton G

Positive instance: G is injective (=bijective=reversible).

Proposition. Decision problem **2D reversibility** is undecidable. It is semi-decidable while the complement is not semi-decidable.

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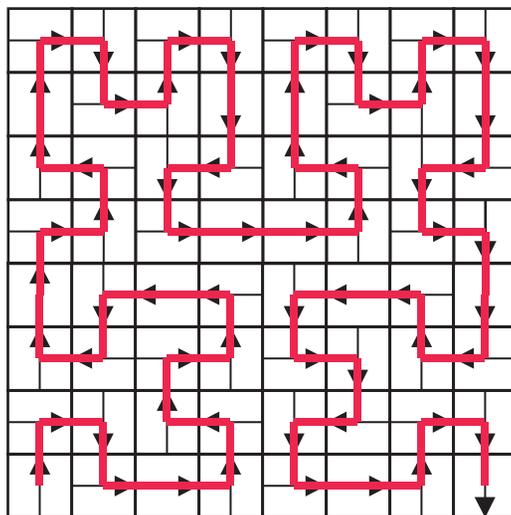
In the reduction we use our specific directed tile set **SNAKES** that satisfies the plane-filling property.

Recall: The **plane-filling property** means that

- **SNAKES** admits a **valid tiling** of the plane, and
- on any configuration (valid tiling or not) and on **any infinite path** that follows the directions one of the following two things happens:

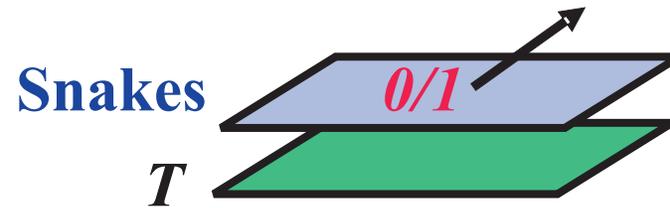
(1) Either there is a **tiling error** between two tiles both of which are on the path,

(2) or the path is a **plane-filling path**, that is, for every positive integer n there exists an $n \times n$ square all of whose positions are visited by the path.



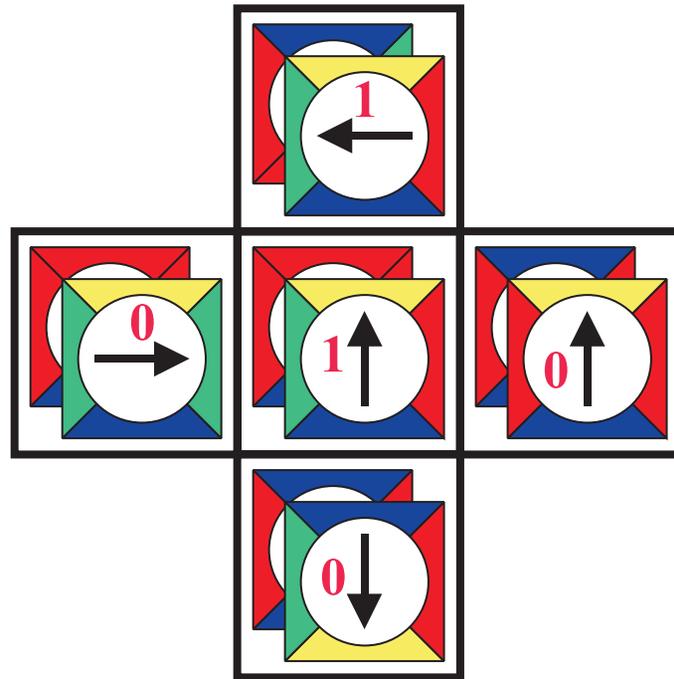
Reduction: For any given tile set T we construct a CA with the state set

$$S = T \times \text{SNAKES} \times \{0, 1\}.$$



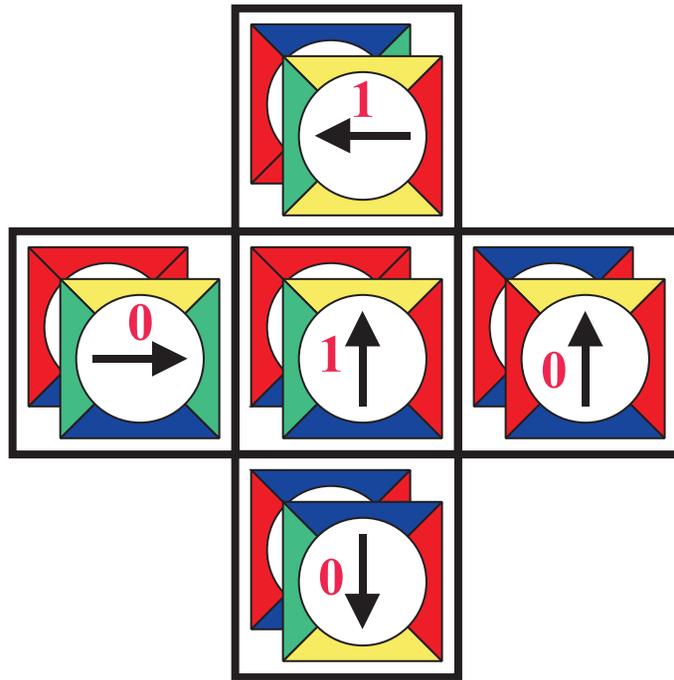
The local rule is analogous to **Snake XOR** with the difference that the correctness of the tiling is checked in both tile layers:

- If there is a tiling error then the cell is inactive.



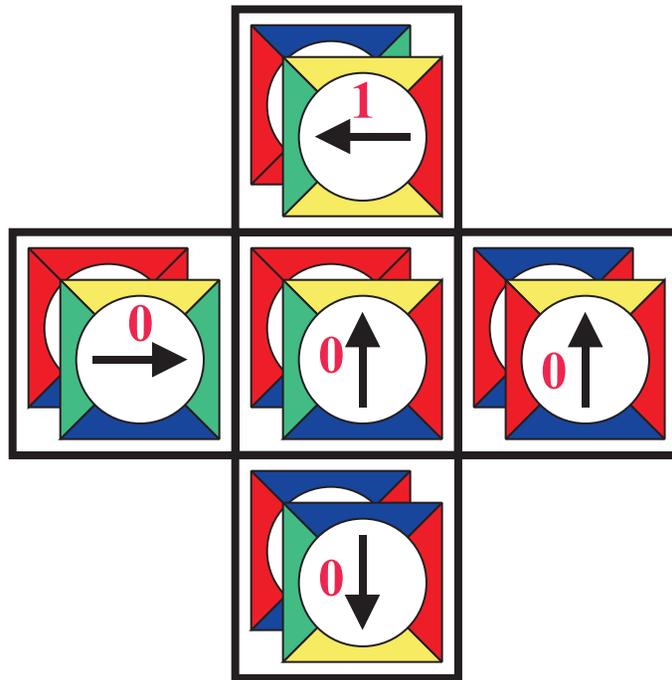
The local rule is analogous to **Snake XOR** with the difference that the correctness of the tiling is checked in both tile layers:

- If there is a tiling error then the cell is inactive.
- If both tilings are valid, the bit of the neighbor next on the path is XOR'ed to the bit of the cell.



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We can reason exactly as with **Snake XOR**, and show that the CA is reversible if and only if the tile set T does not admit a plane tiling:

(\implies) If a valid tiling of the plane exists then we can construct two different configurations of the CA that have the same image under G . The **SNAKES** and the T layers of the configurations form the same valid tilings of the plane. In one of the configurations all bits are 0, and in the other configuration all bits are 1.

All cells are active because the tilings are correct. This means that all bits in both configurations become 0. So the two configurations become identical. The CA is not injective.

(\Leftarrow) Conversely, assume that the CA is not injective. Let c and d be two different configurations with the same successor. Since only bits may change, c and d must have identical **SNAKES** and T layers. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

Because c and d have identical successors:

- The cell in position \vec{p}_1 must be active, that is, the **SNAKES** and T tilings are both valid in position \vec{p}_1 .
- The bits stored in the next position \vec{p}_2 (indicated by the direction) are different in c and d .

Hence we can repeat the reasoning in position \vec{p}_2 .

The same reasoning can be repeated over and over again. The positions $\vec{p}_1, \vec{p}_2, \vec{p}_3, \dots$ form a path that follows the arrows on the tiles. There is no tiling error at any tile on this path so the special property of **SNAKES** forces the path to cover arbitrarily large squares.

Hence T admits tilings of arbitrarily large squares, and consequently a tiling of the infinite plane.

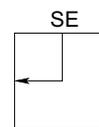
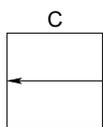
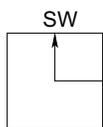
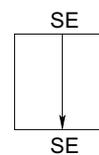
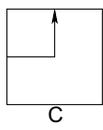
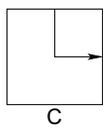
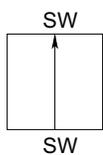
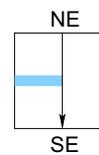
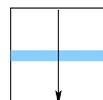
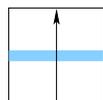
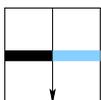
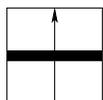
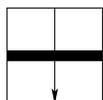
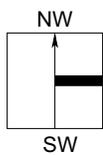
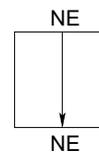
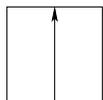
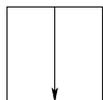
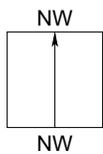
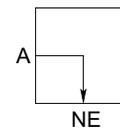
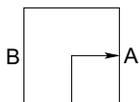
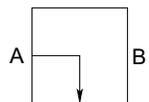
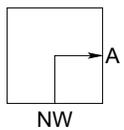
2D surjectivity

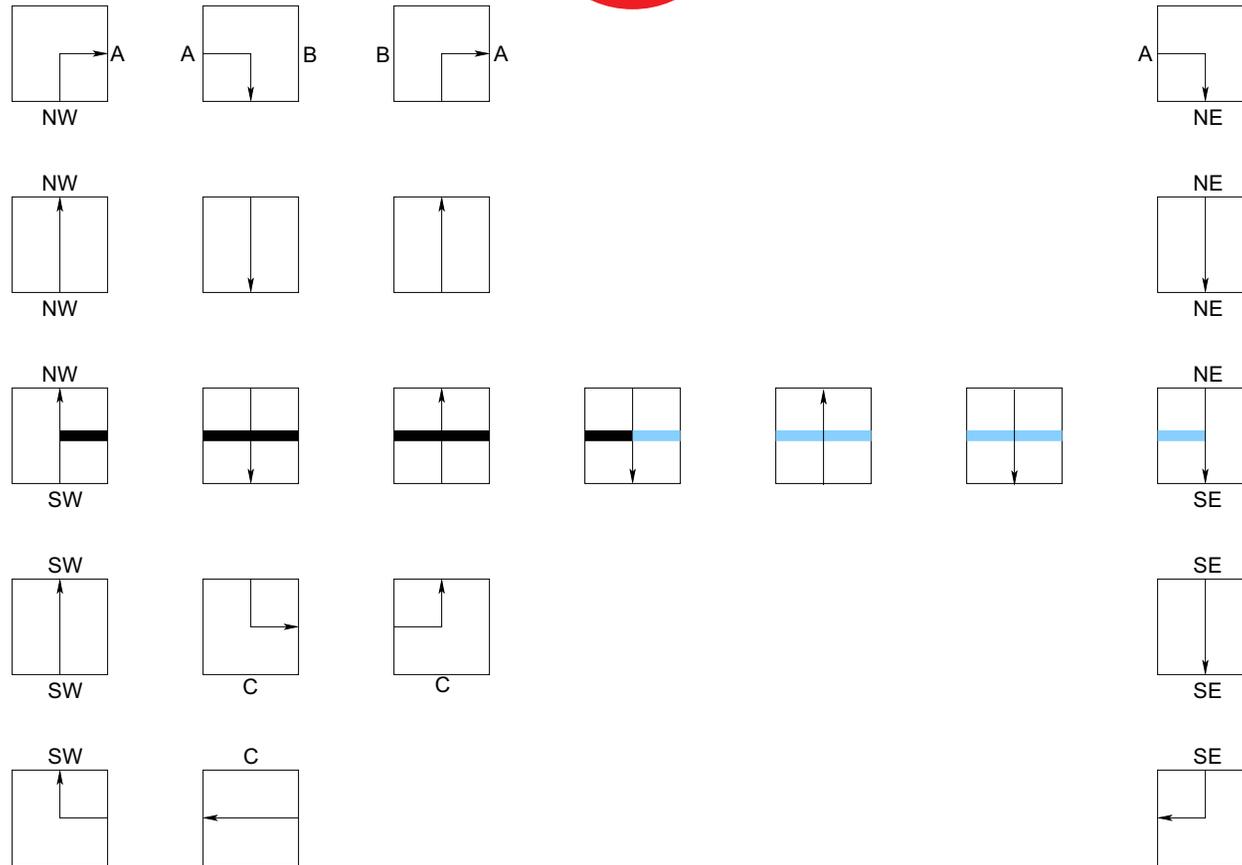
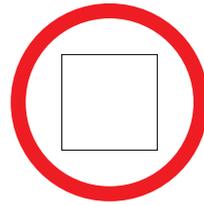
Instance: A two-dimensional cellular automaton G

Positive instance: G is surjective (=pre-injective).

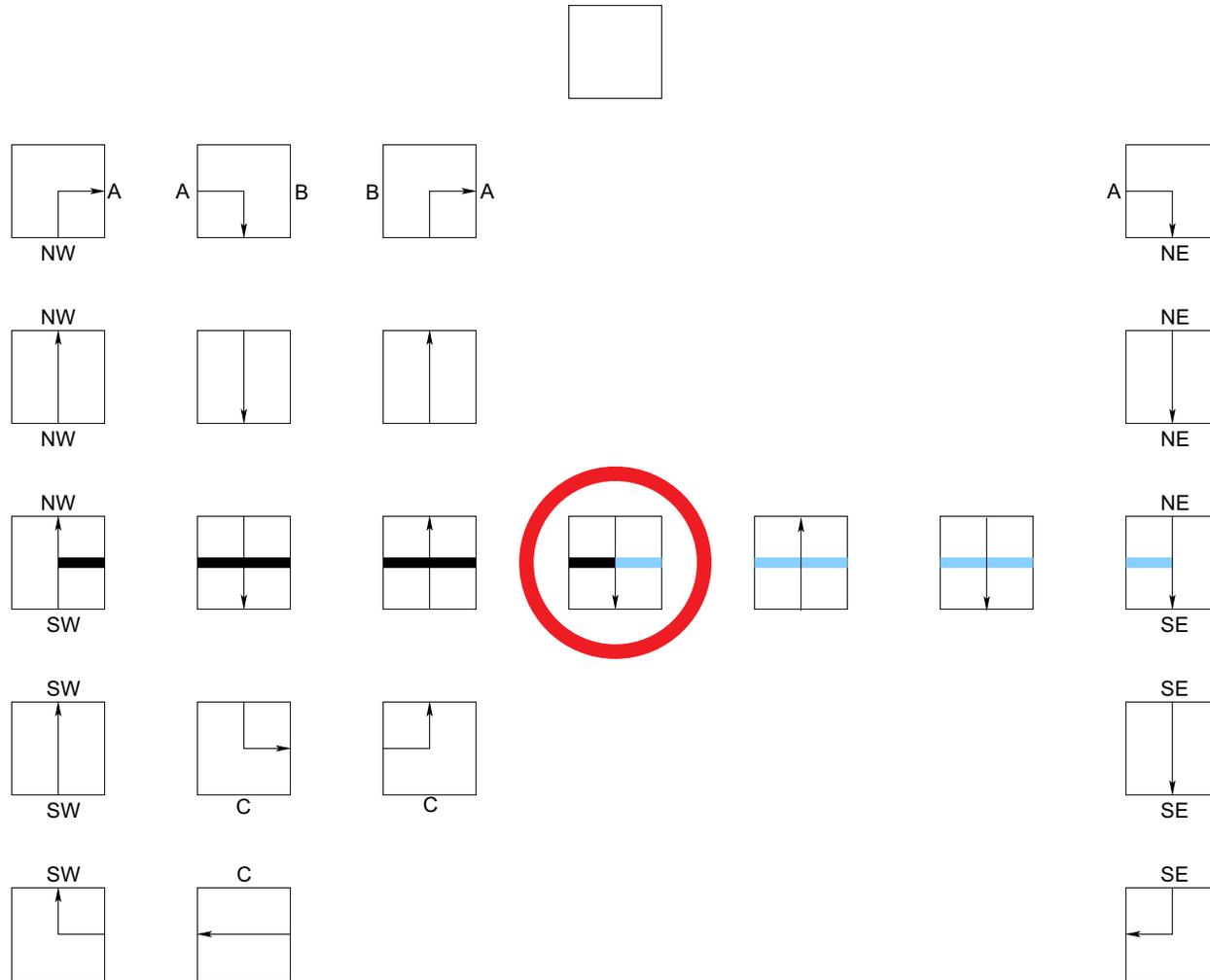
Proposition. Decision problem **2D surjectivity** is undecidable. It is not semi-decidable while the complement is semi-decidable.

Proof. The proof of undecidability is a many-one reduction from **Finite tiling problem**. In the reduction we need a particular set D of 23 Wang tiles.

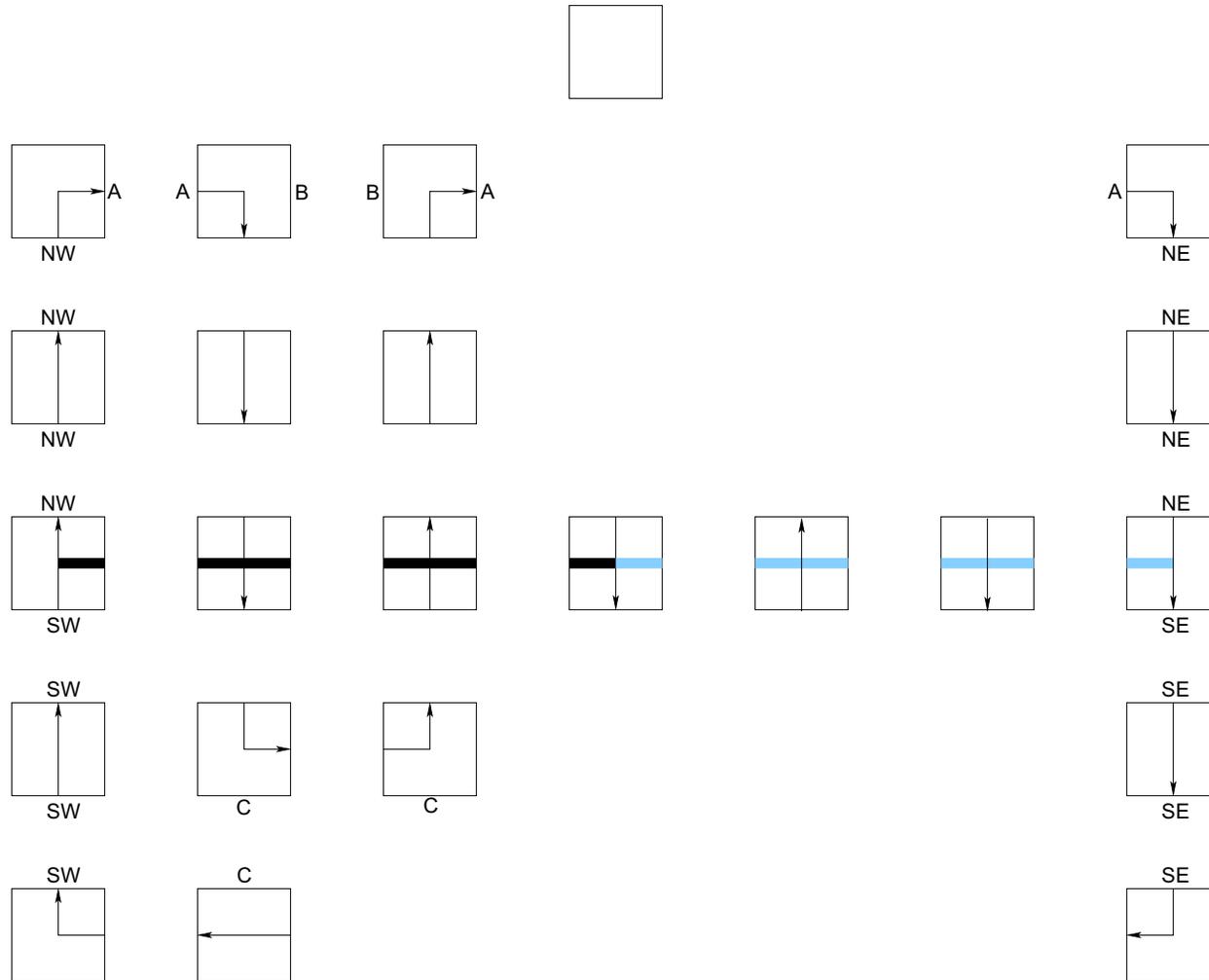




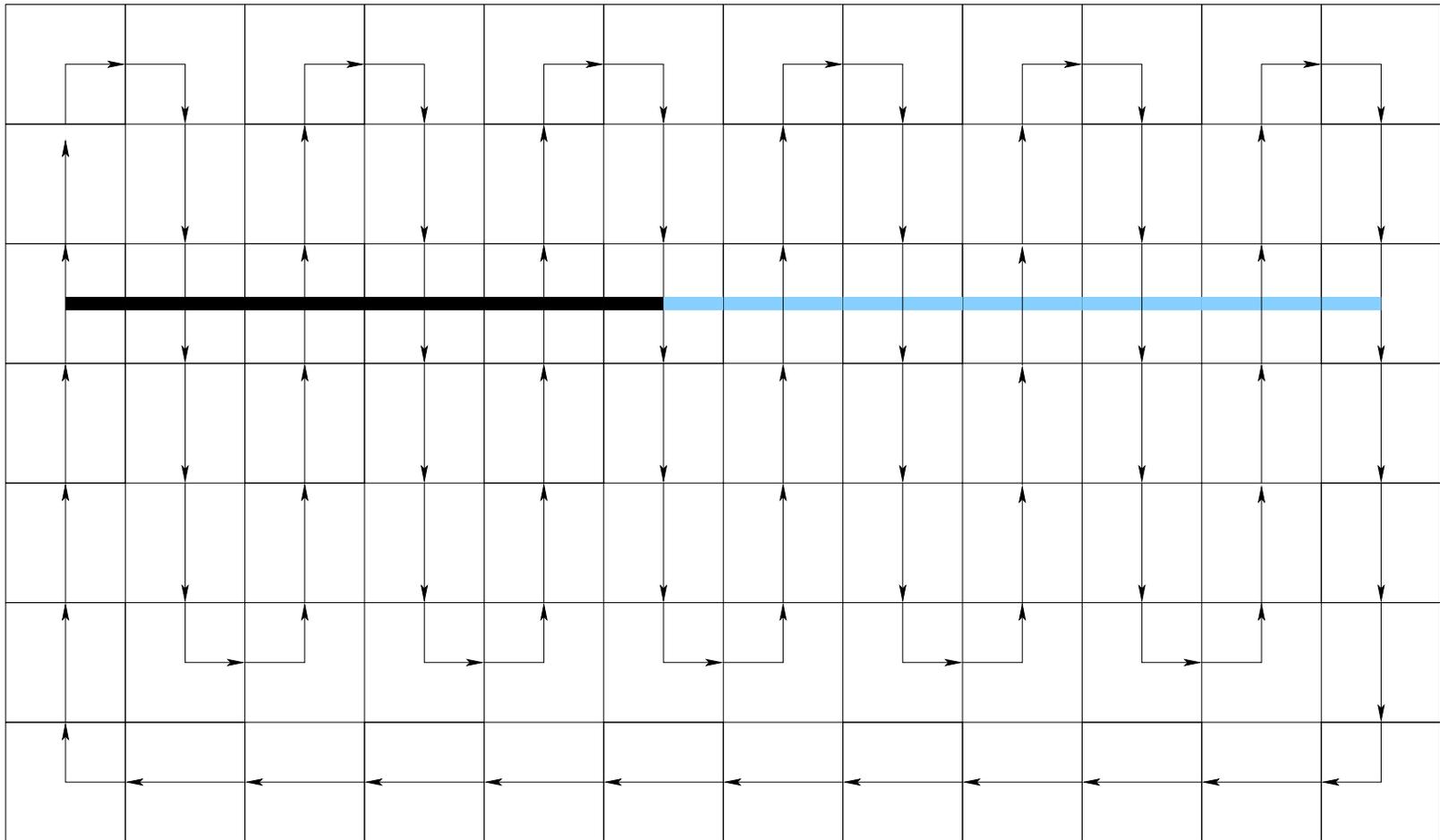
The topmost tile b is a **blank**.



The tile shown at the center is the **cross**.

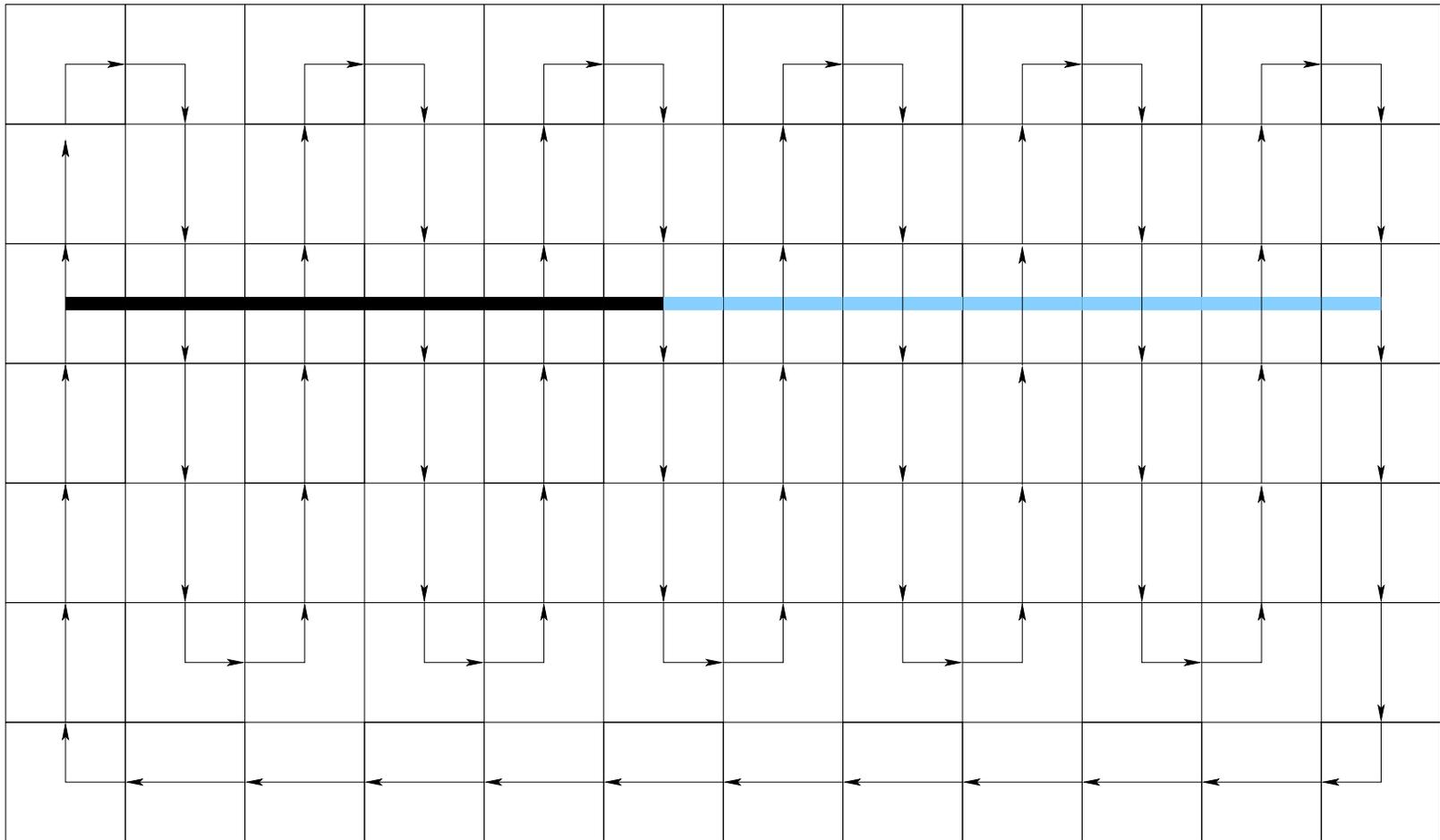


Every non-blank tile is **directed**: it has an arrow pointing to the follower. In valid tilings arrows and labels must match. All non-blank tiles have a unique incoming and outgoing arrow, so if a tiling is valid at a non-blank tile then the tile has a unique follower and a unique precursor.



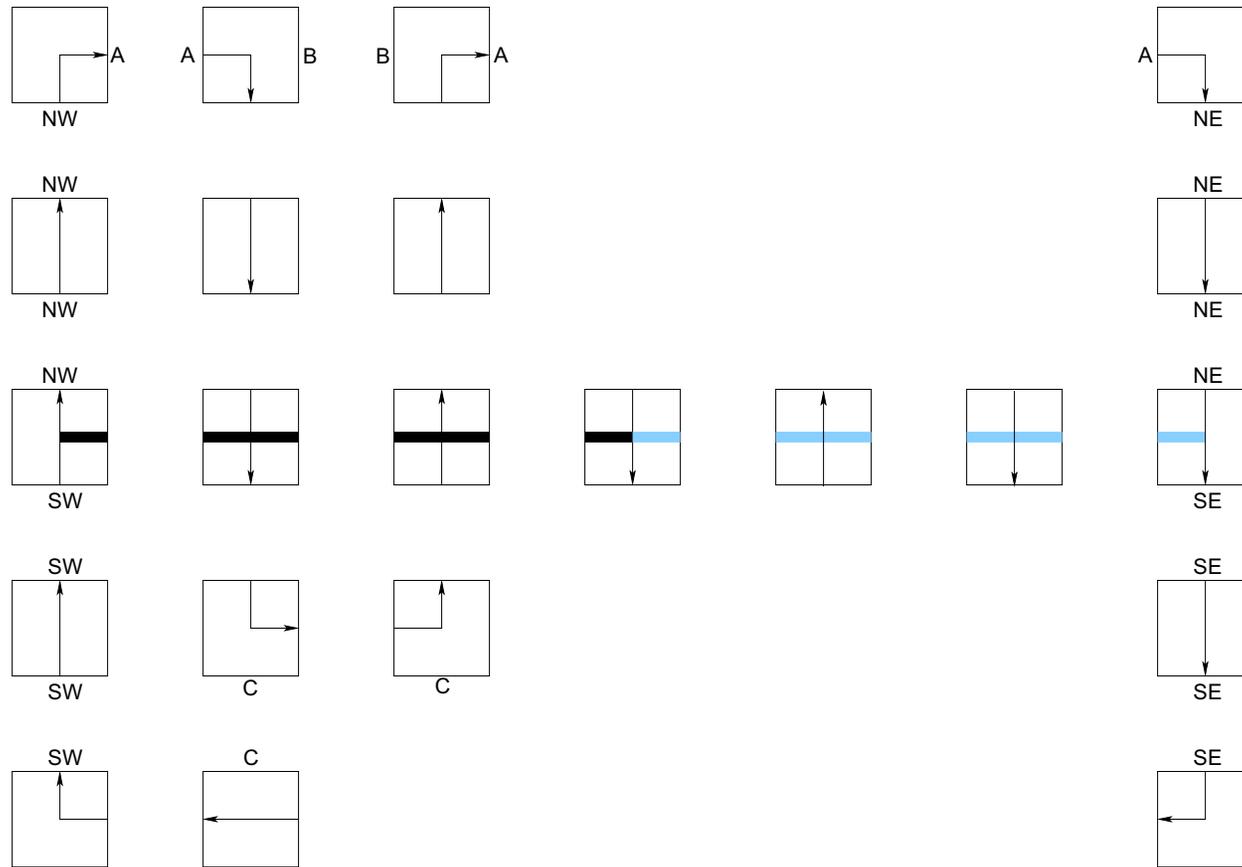
For any even $n \geq 4$ and any $m \geq 4$ the tiles can form a **rectangular loop** with n columns and m rows.

The sample above is a rectangular loop of size 12×7 .



A rectangular loop

- consists a cyclic follower path that visits each tile of the rectangle,
- has only blank colors on the boundaries,
- has exactly one occurrence of the cross in the interior.



Lemma. Let D be the special set of 23 tiles and let $t \in D^{\mathbb{Z}^2}$ be a configuration. Let $\vec{p}_1, \vec{p}_2, \vec{p}_3, \dots$ be a path in t such that the tiling t is valid at \vec{p}_i for all $i = 1, 2, 3, \dots$. If the path covers only a finite number of different cells then the cells on the path form a rectangular loop.

Proof.

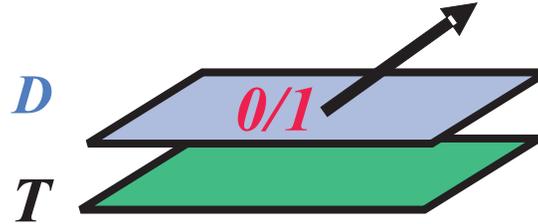
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Let T be any given Wang tile set with a blank tile B . We want to construct a two-dimensional cellular automaton G that is surjective (=pre-injective) if and only if T does not admit a non-trivial B -finite tiling.

Recall that in D we have two special tiles:

- blank b ,
- cross c .



We construct a 2D CA whose states are “sandwich” tiles

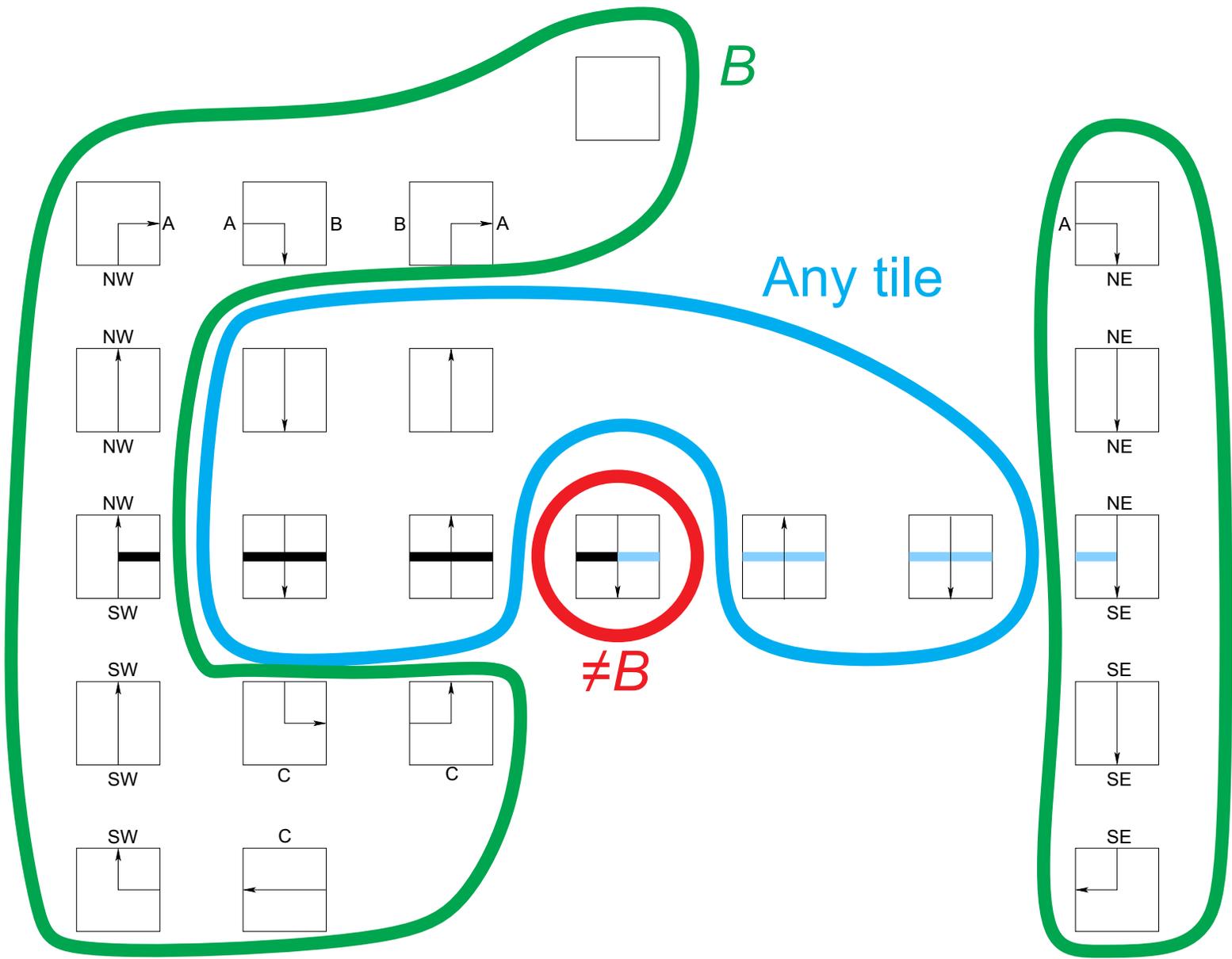
$$(t, d, x) \in T \times D \times \{\mathbf{0}, \mathbf{1}\},$$

but we do not take all such triplets. We limit them as follows:

- If $d = c$ then $t \neq B$, and
- if $d = b$ or d is any tile with a labeled arrow then $t = B$.

Idea: The presence of a cross tile in a rectangular loop forces a non-blank tile in the T -layer, while the boundary tiles of the loop have the blank tile $B \in T$.

Thus a valid tiling on the T -layer of a rectangular loop is possible if and only if T admits a non-trivial valid B -finite tiling.



The local rule: Let (d, t, x) be the current state of a cell.

- If $d = b$ then the state is not changed.
- If $d \neq b$ then the cell verifies the validity of the tilings according to both D and T at the cell.
 - (a) If both tilings are valid then the cell adds to its bit (mod 2) the bit of its follower.
 - (b) If either tile component has a tiling error then the state is not changed.

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This CA is not surjective if and only if T admits a valid, B -finite, non-trivial tiling.

Proof.