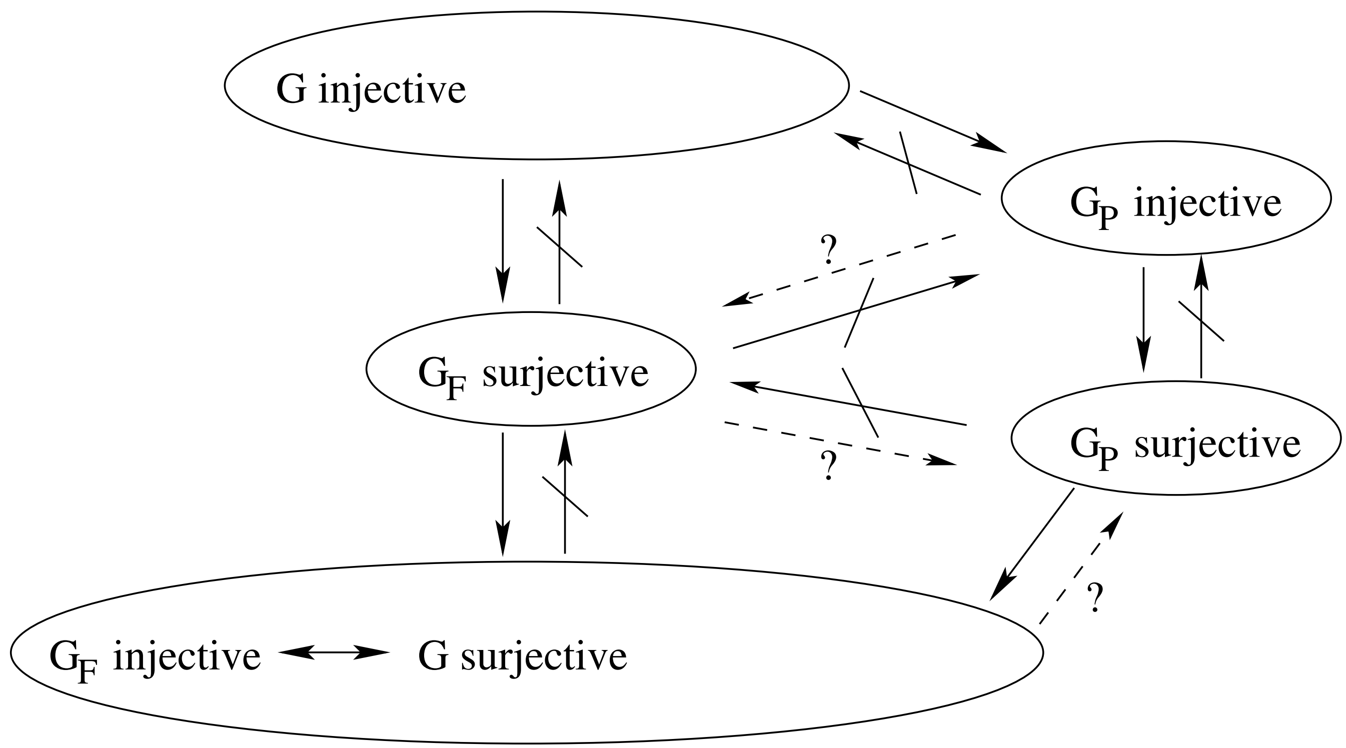
Two-dimensional CA and tilings

We have proved all positive implications in the diagram for dimensions $d \geq 2$:



Example. Recall the 1D CA with states $S = \{0, 1, 2\}$, radius-$\frac{1}{2}$ neighborhood, local rule

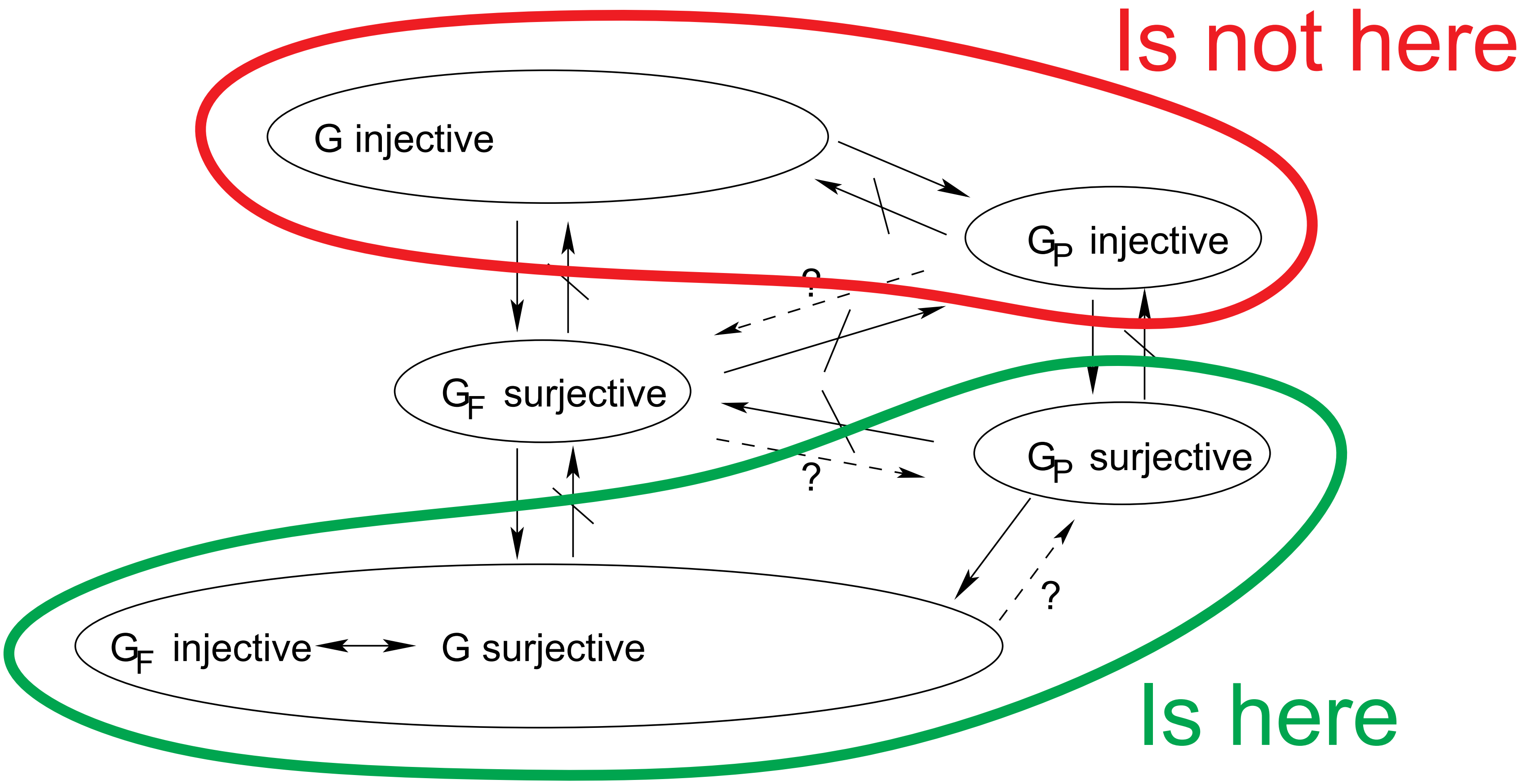
$$f(a,b) = \begin{cases} 2, & \text{if } a = 2, \\ 0, & \text{if } a \neq 2 \text{ and } a+b \text{ is even, and} \\ 1, & \text{if } a \neq 2 \text{ and } a+b \text{ is odd.} \end{cases}$$

(State 2 is unchanged. States 0 and 1 are changed using modulo 2 sum with the right neighbor.)

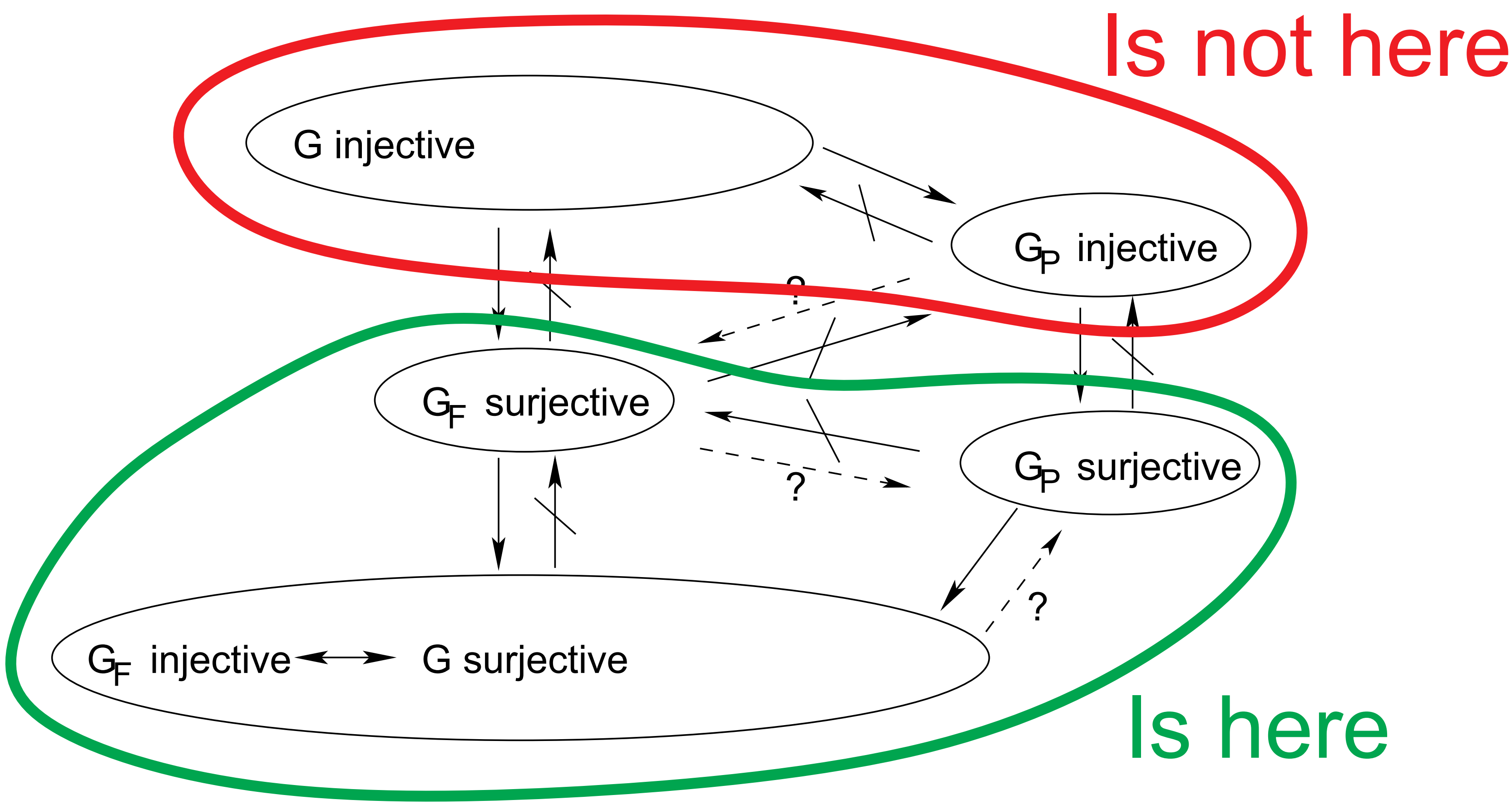
Apply this CA on **horizontal rows** of 2D configurations (or in the general d-dimensional case on lines in the first coordinate direction, independently of each other).

Then G_P **is surjective** but G_P **is not injective**.

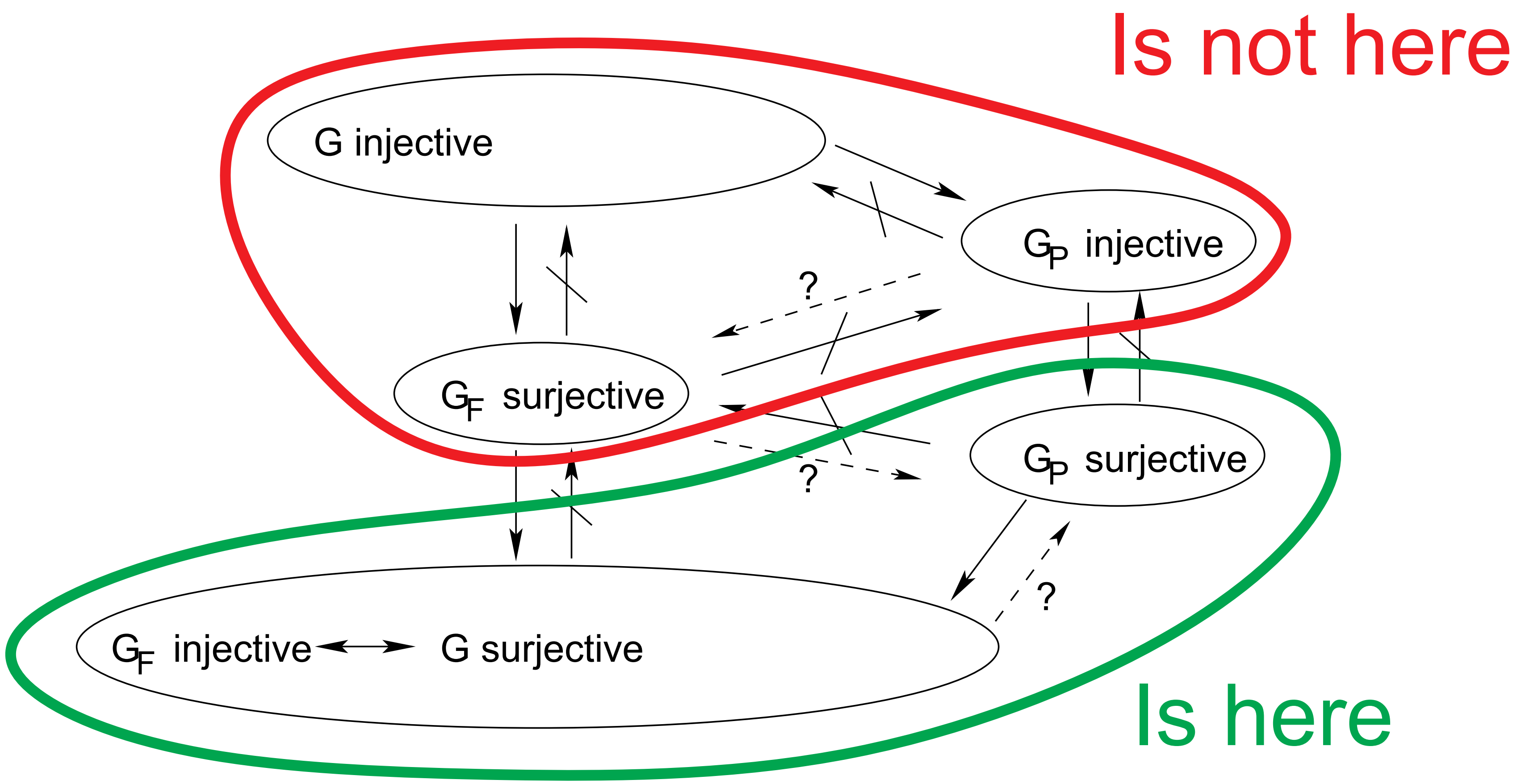
G_P is surjective but G_P is not injective:



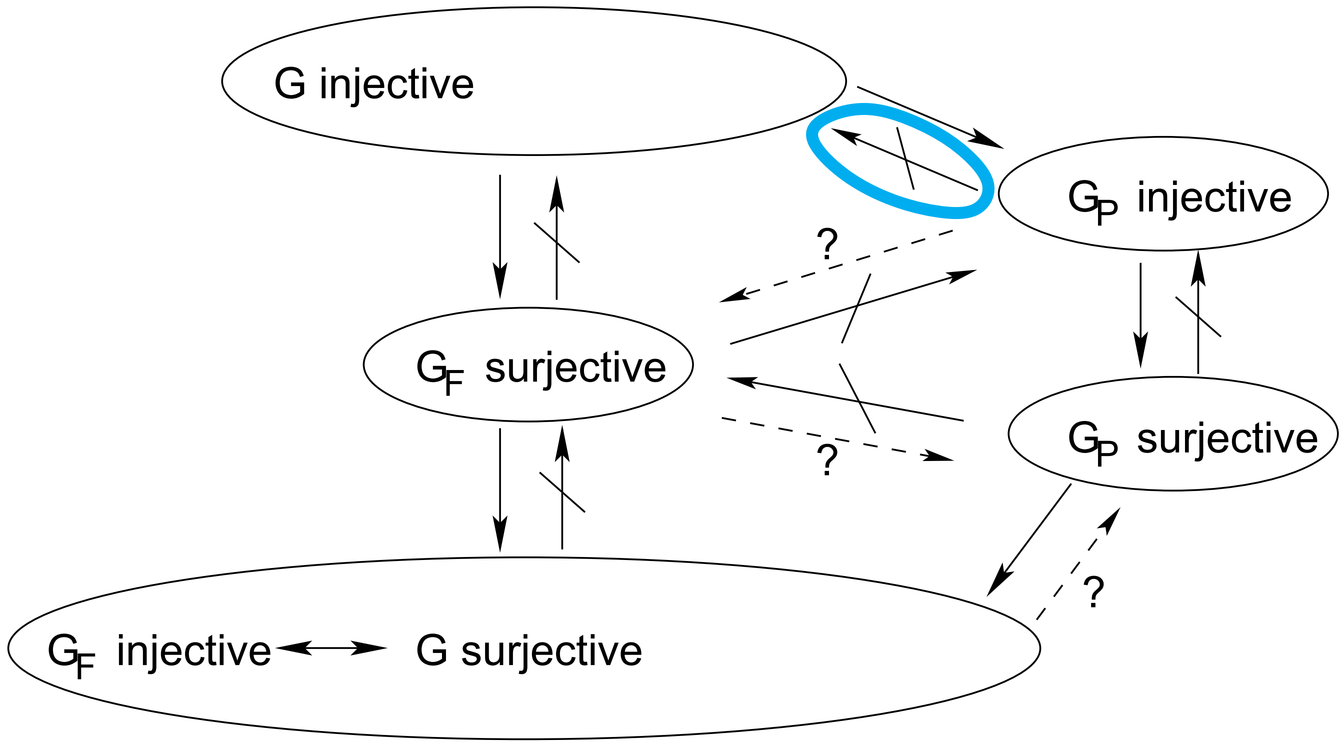
If state $q = 2$ is the quiescent state then G_F is surjective:



If state $q = 0$ is the quiescent state then G_F is not surjective:



The example gave the negative implications except for one:



We use **aperiodic tiles** in an example of a 2D CA that is not injective but that is injective on strongly periodic configurations.

Definition. A **tile set**

$$\mathcal{T} = (T, N, R)$$

consists of

- a finite set T of **tiles**,
- a two-dimensional **neighborhood vector**

  $$N = (\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_m)$$

  where each $\vec{n}_i \in \mathbb{Z}^2$ and $\vec{n}_i \neq \vec{n}_j$ for all $i \neq j$.
- a **local matching rule** $R \subseteq T^m$ specifying which pattern are allowed in valid tilings.

Definition. A **tile set**

$$\mathcal{T} = (T, N, R)$$

consists of

- a finite set T of **tiles**,
- a two-dimensional **neighborhood vector**

  $$N = (\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_m)$$

  where each $\vec{n}_i \in \mathbb{Z}^2$ and $\vec{n}_i \neq \vec{n}_j$ for all $i \neq j$.
- a **local matching rule** $R \subseteq T^m$ specifying which pattern are allowed in valid tilings.

Definition. **Tilings** are configurations $t \in T^{\mathbb{Z}^2}$. A tiling t is **valid at cell** $\vec{n} \in \mathbb{Z}^2$ if and only if

$$[t(\vec{n} + \vec{n}_1), t(\vec{n} + \vec{n}_2), \ldots, t(\vec{n} + \vec{n}_m)] \in R.$$

Tiling $t \in T^{\mathbb{Z}^2}$ is **valid** if it is valid at all positions $\vec{n} \in \mathbb{Z}^2$.

Let $V(\mathcal{T})$ be **the set of all valid tilings** admitted by $\mathcal{T}$.

Definitions of cellular automata and tile sets are very similar. Difference:

- Cellular automata have **dynamic** update function,
- Tile sets have **static** matching relation

In symbolic dynamics terminology, $V(\mathcal{T})$ is a two-dimensional **subshift of finite type (SFT)**.

CA have two fundamental properties: they **commute with translations** and are **continuous**. Correspondingly, the set $V(\mathcal{T})$ of valid tilings has the fundamental properties that it is

- **invariant under translations**, and
- **closed** (limit of a converging sequence of valid tilings is also valid).

Proposition. Let $\mathcal{T} = (T, N, R)$ be a tile set.

(i) If t is a valid tiling and τ is a translation of the plane then $\tau \circ t$ is a valid tiling.

(ii) Suppose $t_1, t_2, \ldots$ is a converging sequence where for every $\vec{n} \in \mathbb{Z}^2$ there is k such that t_i is valid at cell $\vec{n}$ for all $i \geq k$. Then $\lim_{i\to\infty} t_i$ is a valid tiling.

Proof.

CA have two fundamental properties: they **commute with translations** and are **continuous**. Correspondingly, the set $V(\mathcal{T})$ of valid tilings has the fundamental properties that it is

- **invariant under translations**, and
- **closed** (limit of a converging sequence of valid tilings is also valid).

Proposition. Let $\mathcal{T} = (T, N, R)$ be a tile set.

(i) If t is a valid tiling and τ is a translation of the plane then $\tau \circ t$ is a valid tiling.

(ii) Suppose $t_1, t_2, \ldots$ is a converging sequence where for every $\vec{n} \in \mathbb{Z}^2$ there is k such that t_i is valid at cell $\vec{n}$ for all $i \geq k$. Then $\lim_{i\to\infty} t_i$ is a valid tiling.

As a consequence of (ii), if a tile set admits tilings of arbitrarily large regions then it admits a tiling of the whole infinite plane:

Corollary. Suppose that for every finite $D \subseteq \mathbb{Z}^2$ there is $t \in T^{\mathbb{Z}^2}$ that is valid at every $\vec{n} \in D$. Then there is a valid tiling of the plane.

Proof.

Another property of two-dimensional tilings:

Proposition. If $V(\mathcal{T})$ contains a periodic configuration then it also contains a strongly periodic configuration.

Proof.