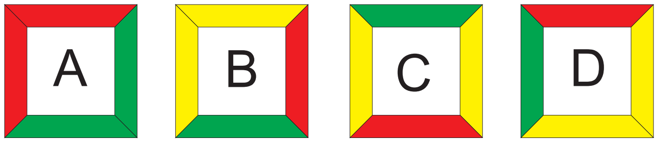
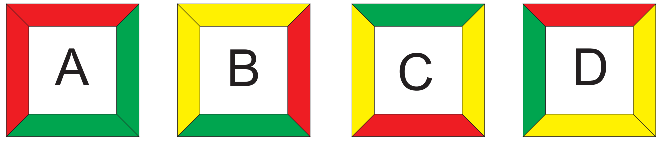
Wang tiles are a popular way to describe tilings. Wang tiles use the von Neumann neighborhood. The tiles are unit squares whose edges are colored:



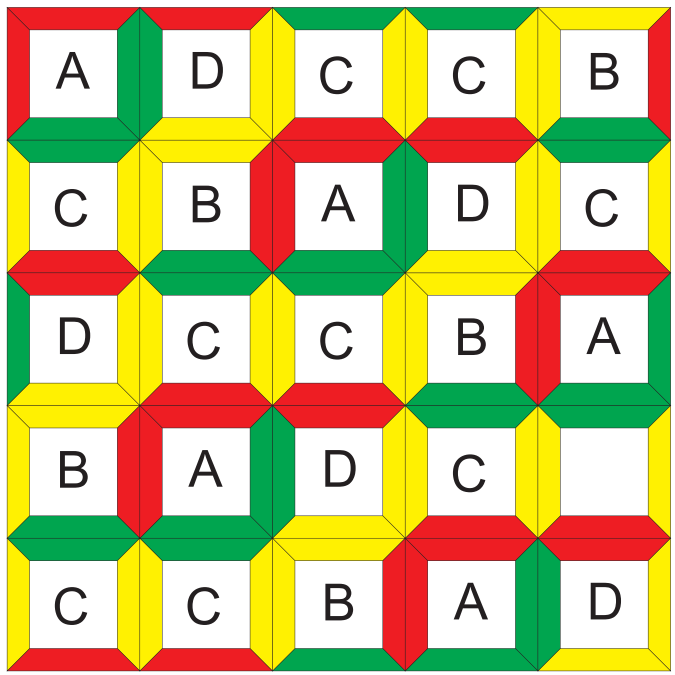
The local matching rule is given in terms of the colors: A tiling is valid at cell $\vec{n} \in \mathbb{Z}^2$ iff each of the four edges of the tile in position $\vec{n}$ have the same color as the abutting edge in the adjacent tile.

Tiles may not be rotated – used in the given orientation.

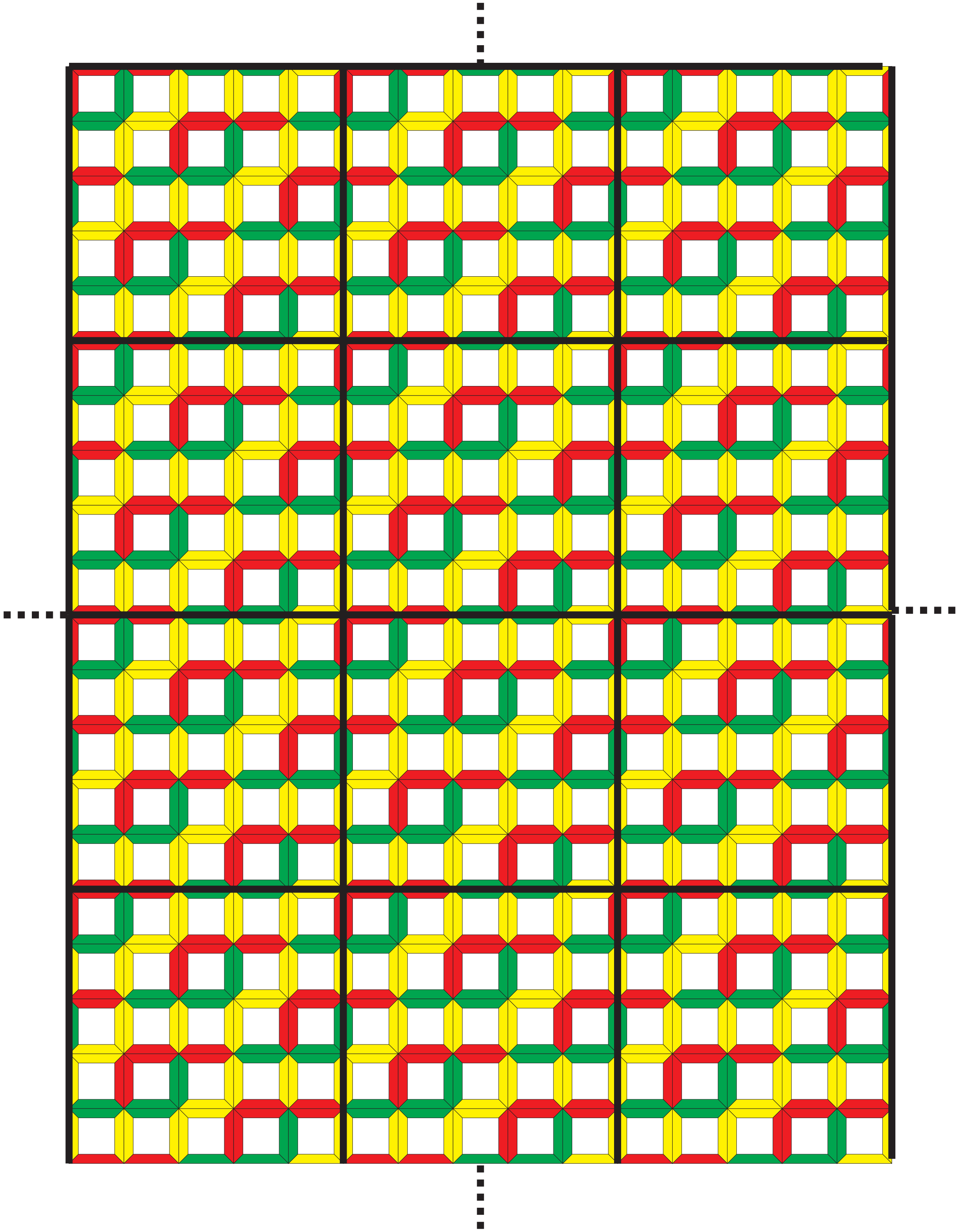
Example. With



we can tile:



. . . and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the whole plane.



A tile set is called **aperiodic** if

(i) it admits some valid tilings ($V(\mathcal{T}) \neq \emptyset$) but

(ii) it does not admit valid periodic tilings (no periodic configurations in $V(\mathcal{T})$).

It was believed that aperiodic tile sets do not exist. This belief was refuted in 1964 by R.Berger who constructed a tile set that enforces non-periodicity.

Proposition. There exist aperiodic sets of Wang tiles.

Proof. Skipped. (Proved on the course "Tilings and Patterns".)

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Proof. Skipped. (Proved on the course "Tilings and Patterns".)

Berger's first aperiodic set contained **20426** tiles.

Later much smaller sets were found. The smallest aperiodic Wang tile set contains **11** tiles found by E. Jeandel and M. Rao, 2015. They also proved that no smaller aperiodic sets exist.

Example 1. Let $\mathcal{T} = (T, N, R)$ be an **aperiodic Wang tile set**. Construct the following 2D CA:

- State set is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,

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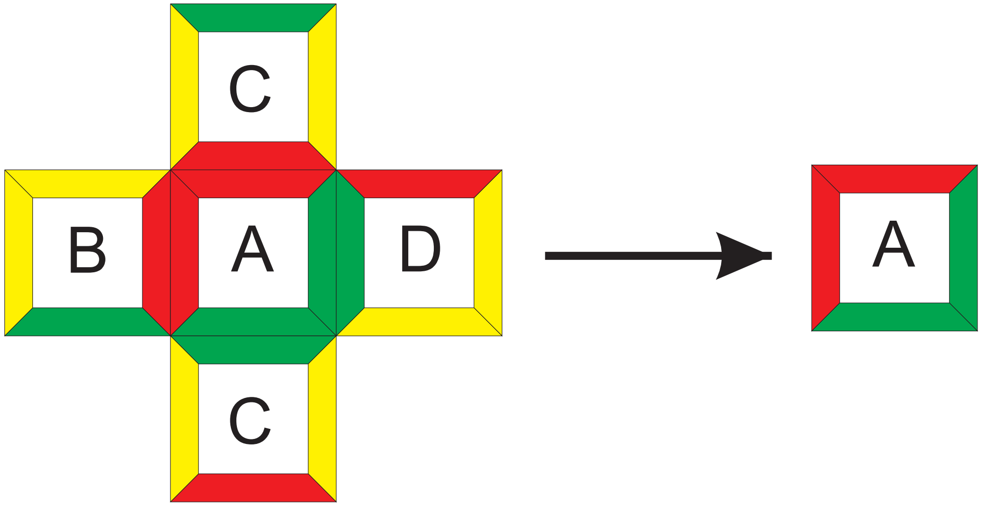

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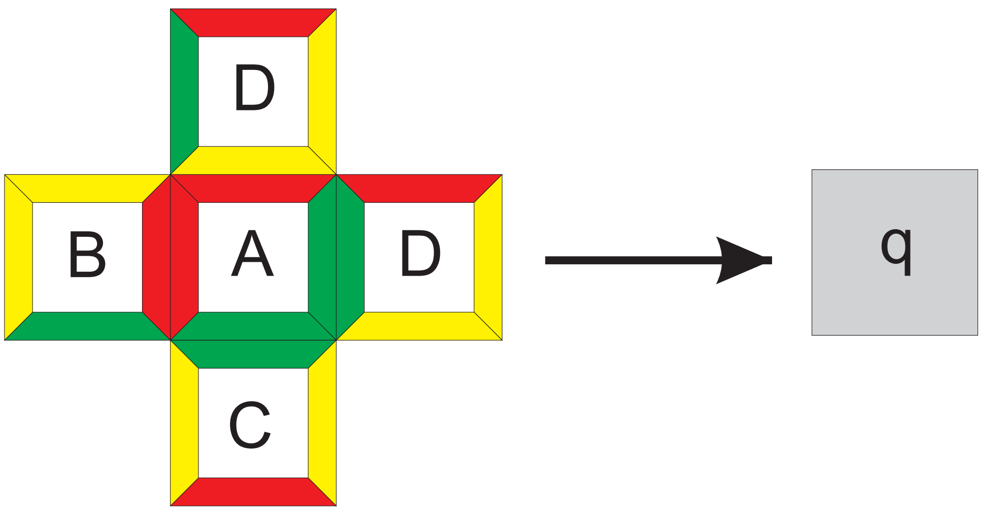

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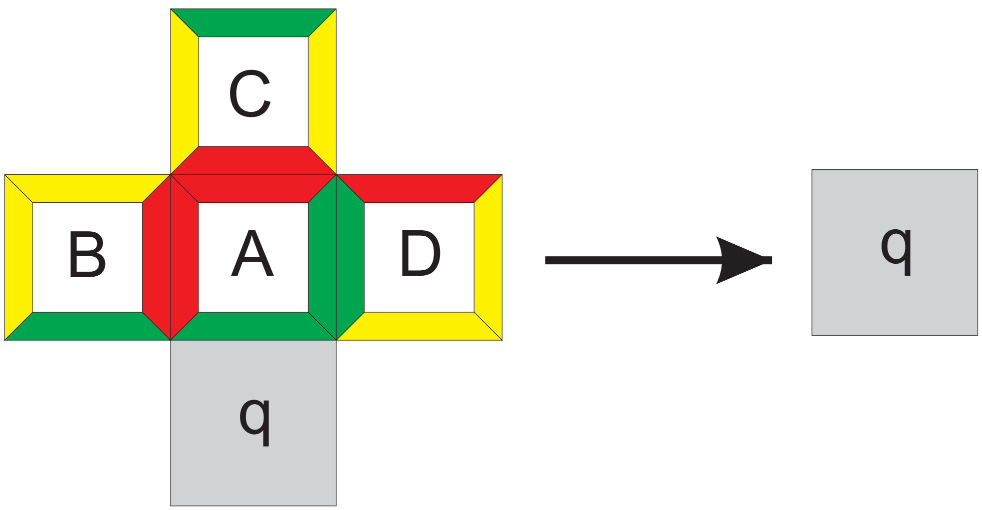

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Properties:

(1) Every strongly periodic configuration becomes eventually quiescent.

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Properties:

(1) Every strongly periodic configuration becomes eventually quiescent.

(2) Not all configurations become quiescent: any valid (non-periodic) tiling is a fixed point.

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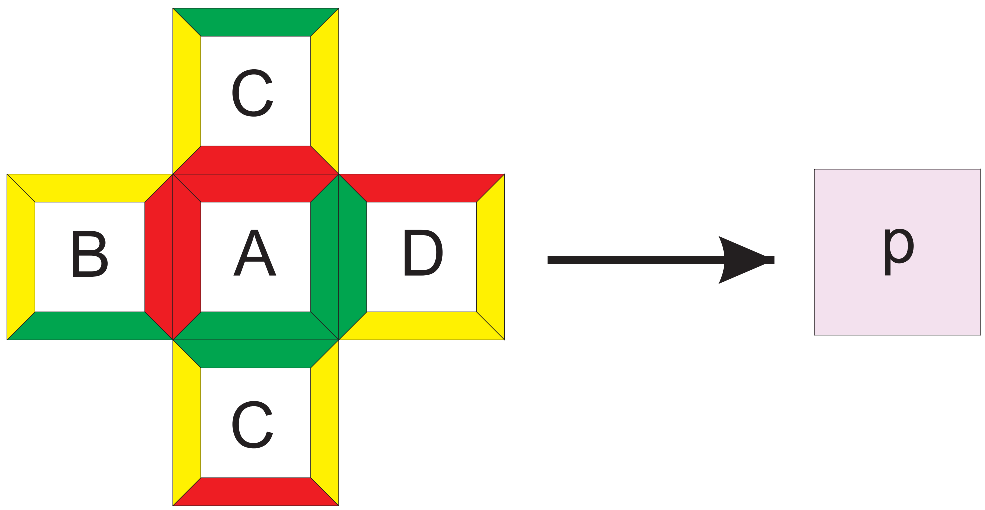

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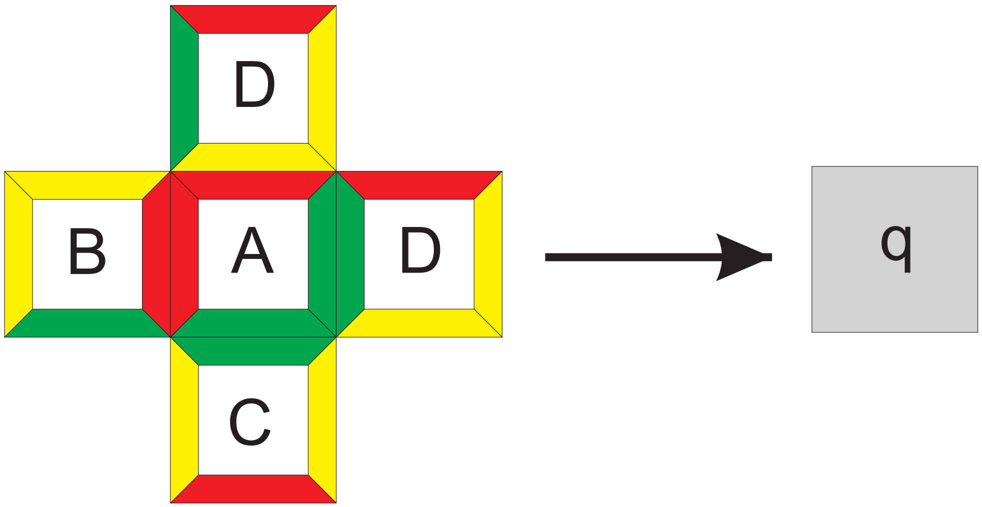

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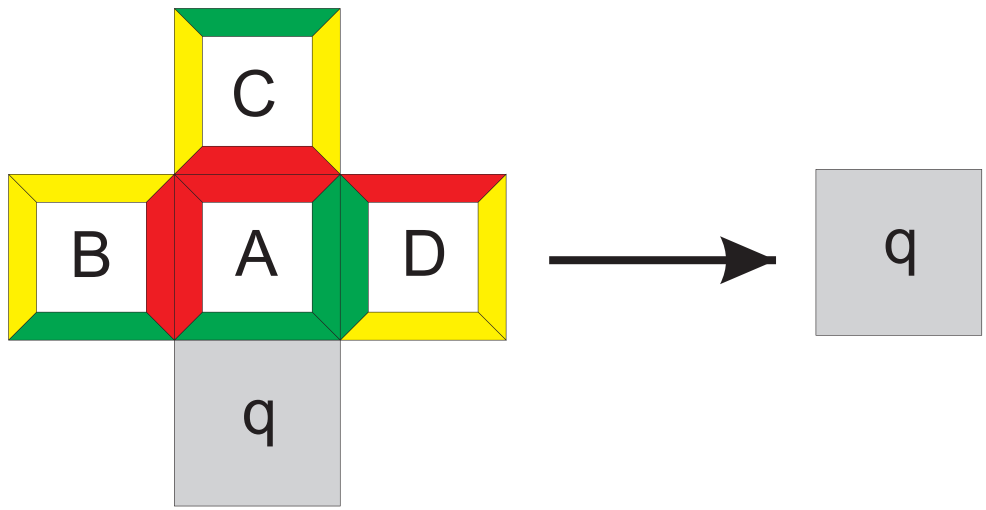

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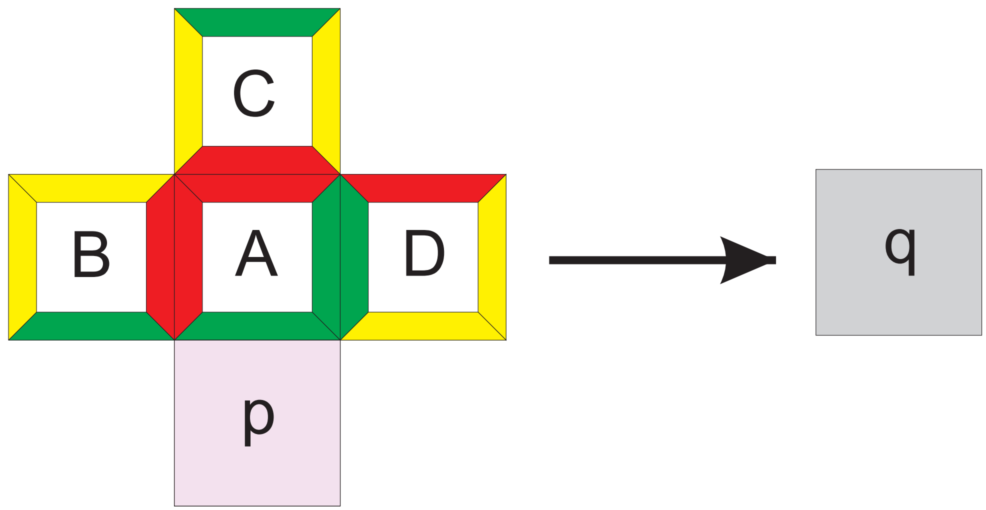

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Property (that cannot happen in 1D):

- There is a strongly periodic configuration (uniform p) that only has non-periodic pre-images.

Example 3. Yet another modification. Still two new states p and q.

The local rule: State p always becomes q, and q always becomes p. If all states in the neighborhood are tiles with matching colors the the state does not change. In all other cases the new state is q.

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Property (that cannot happen in 1D):

- The CA has fixed points, but all the fixed points are non-periodic.

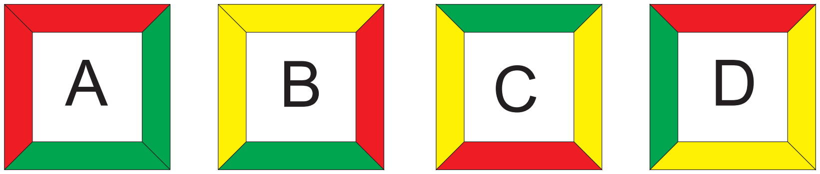
Tiles naturally relate to 2D CA.

However, one can also use tiles to construct interesting **one-dimensional** CA by viewing their **space-time diagrams** as tilings.

A Wang tile set is **NW-deterministic** if no two tiles have identical colors on their top edges and on their left edges. In a valid tiling the left and the top neighbor of a tile uniquely determine the tile.

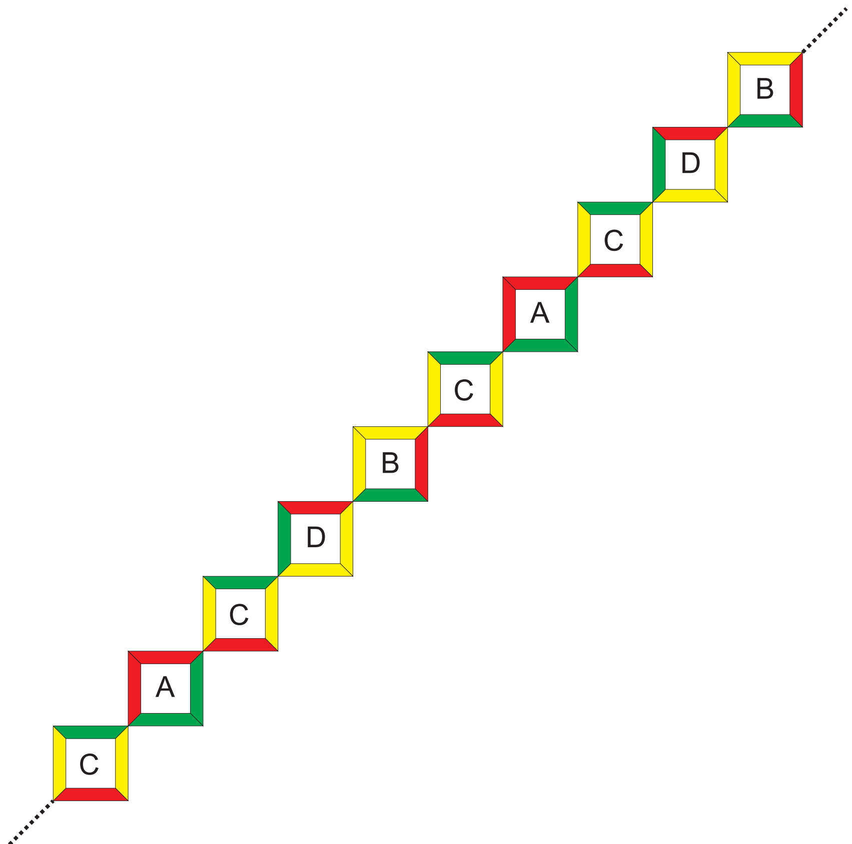
We define analogously NE-, SW- and SE-deterministic tile sets.

For example, our sample tile set



is NW-deterministic.

In any valid tiling by NW-deterministic tiles, NE-to-SW diagonals uniquely determine the next diagonal below them. The tiles of the next diagonal are determined locally from the previous diagonal:



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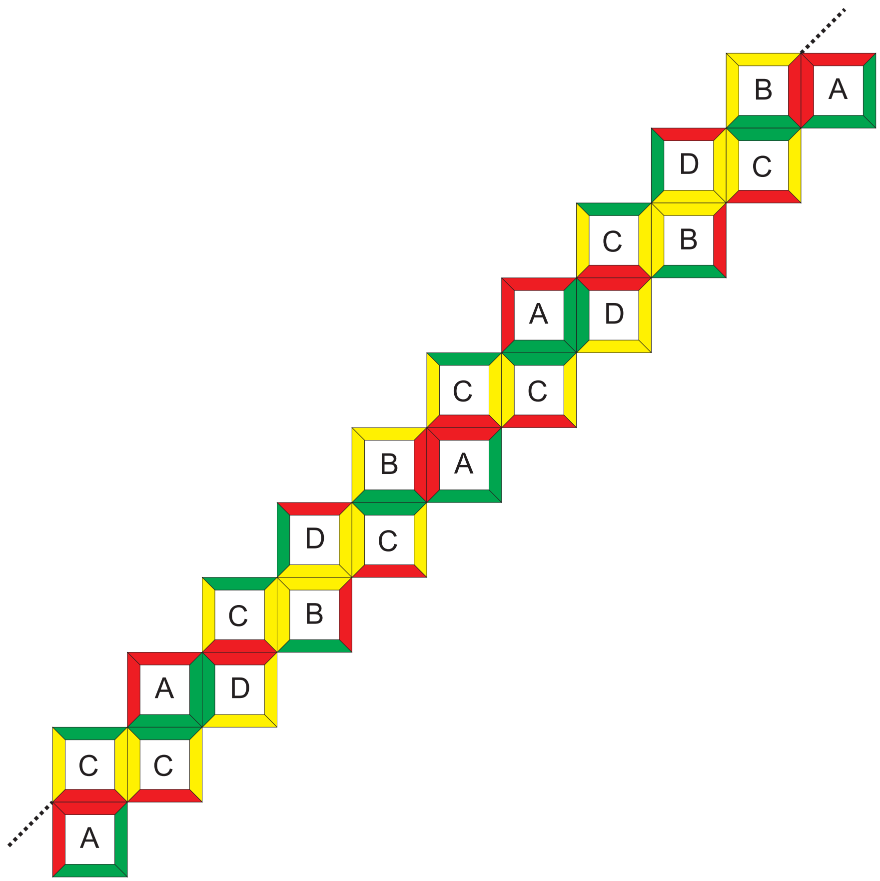

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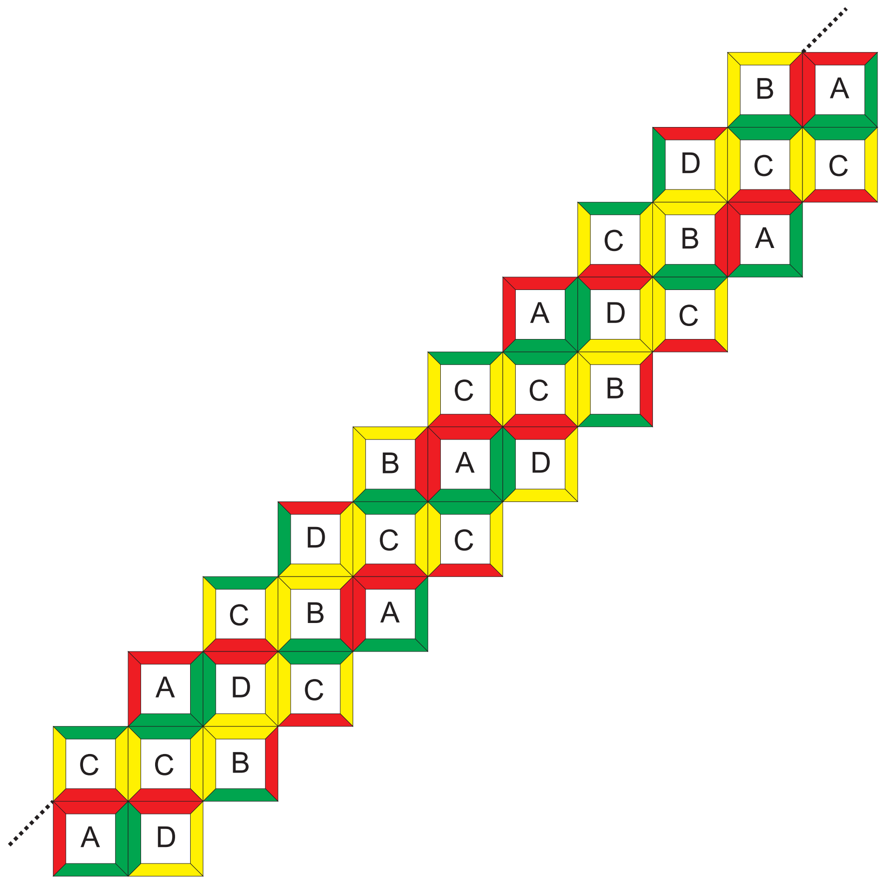

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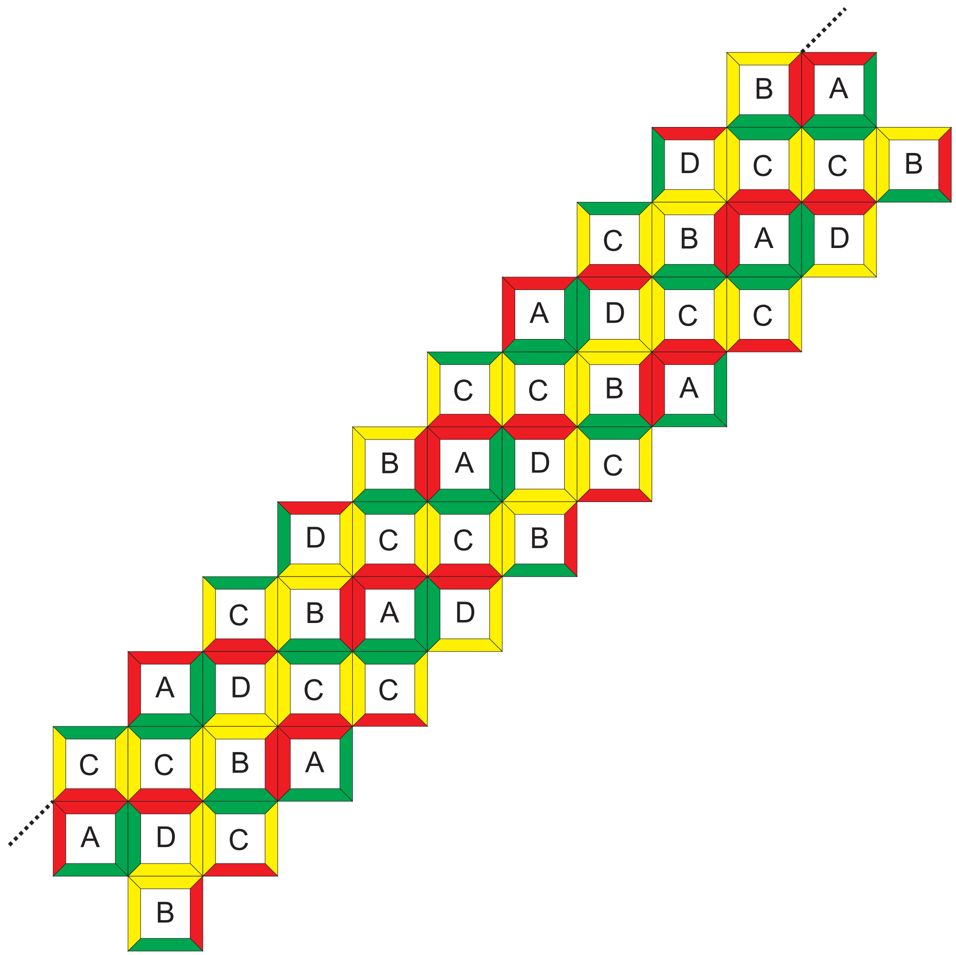

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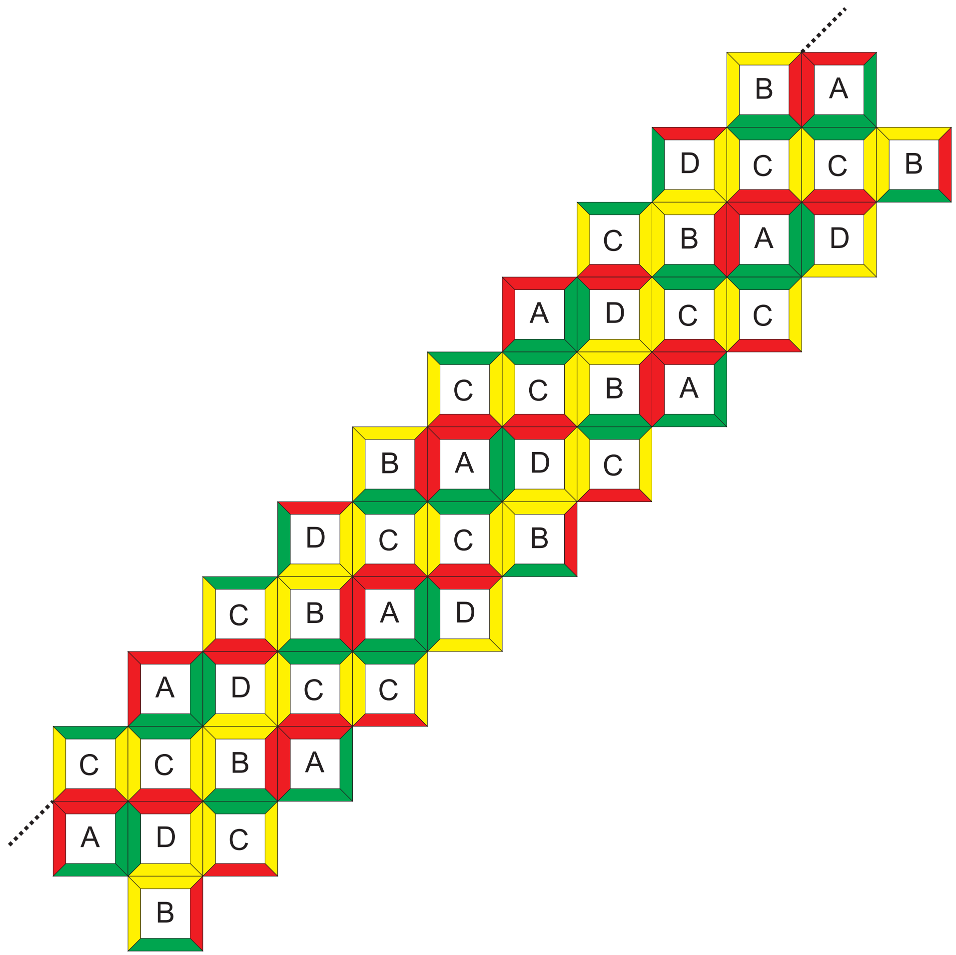
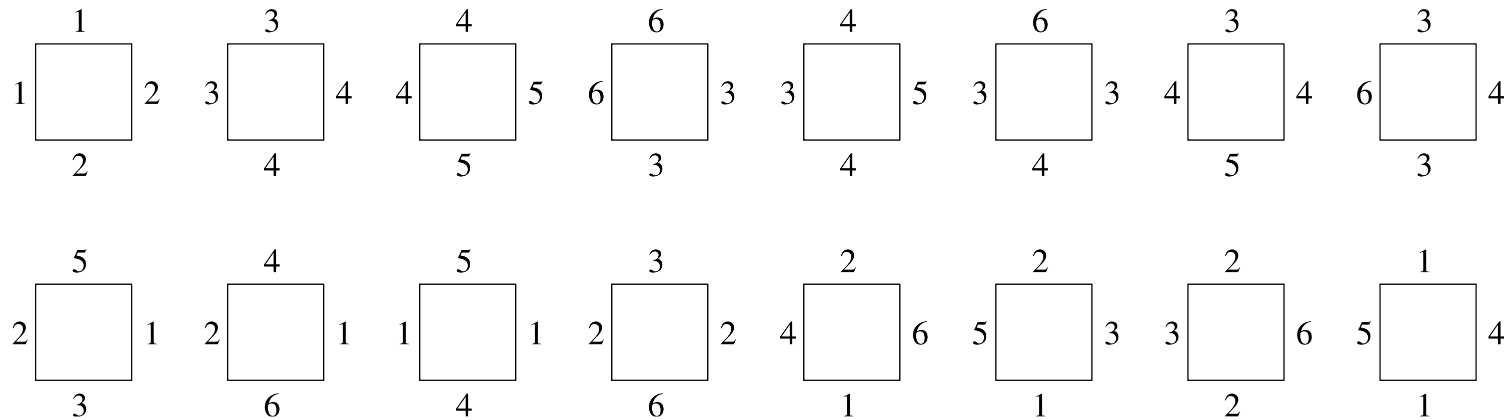


If diagonals are interpreted as configurations of a one-dimensional CA, valid tilings represent space-time diagrams.

There exists NW-deterministic Wang tile sets that are aperiodic. For example, **Amman's aperiodic tile set** from 1977:



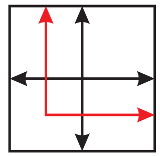
(This was the smallest known aperiodic Wang tile set back then.)

There even exist aperiodic tile sets that are deterministic simultaneously in all four cornerwise directions (proof skipped):

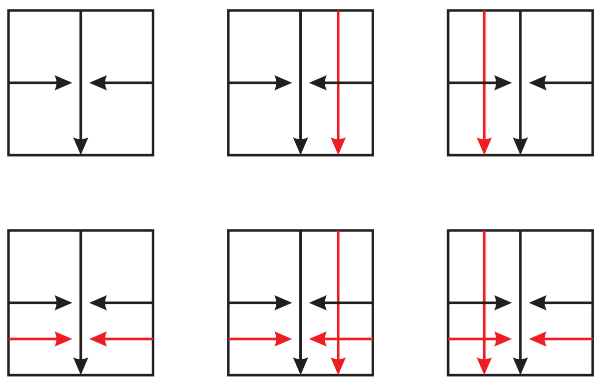
Proposition. There is an aperiodic set of Wang tiles that is NW-, NE-, SW- and SE-deterministic. □

Example. A NW-deterministic version of **Robinson's aperiodic tile set**.

Recall that Robinson tile set consists of **crosses**

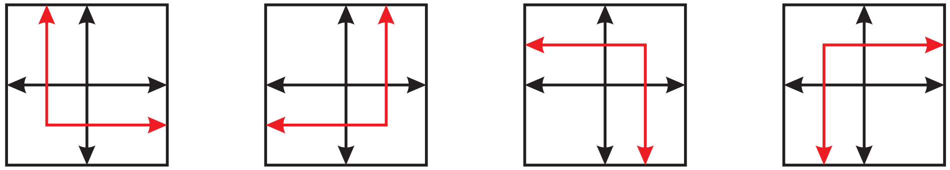


and **arms**

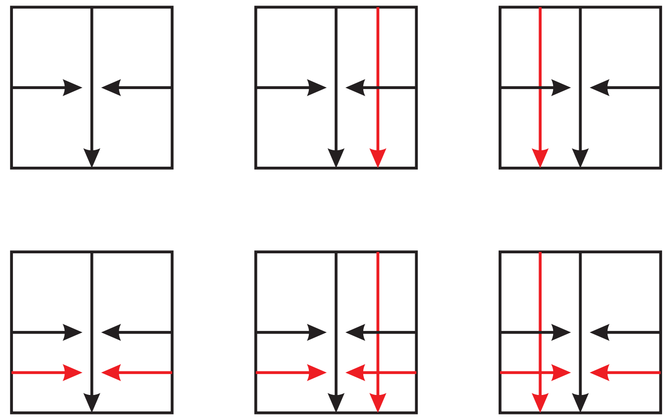


Each tile has a black (incoming or outgoing) arrow on each side, and possibly some red **side arrows**.

The matching rule is that arrows must continue across tile boundaries. Tiles may be rotated freely.



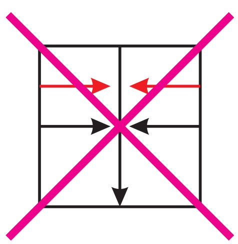
Each cross has two side arrows and the direction of these side arrows determines the **orientation** of the cross.



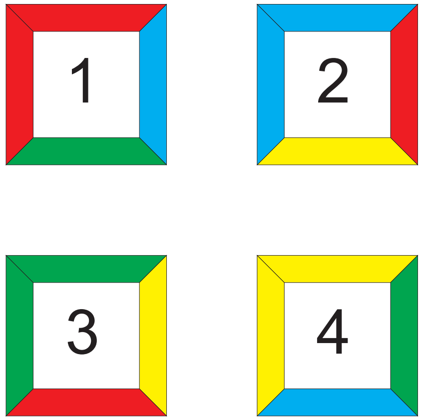
The black arrow through an arm tile is its **principal arrow** and it indicates the orientation of the arm.

Any black arrow may be accompanied by a red side arrow, on either side, with the following exception:

- A pair of incoming side arrows cannot be towards the tail of the principal arrow:



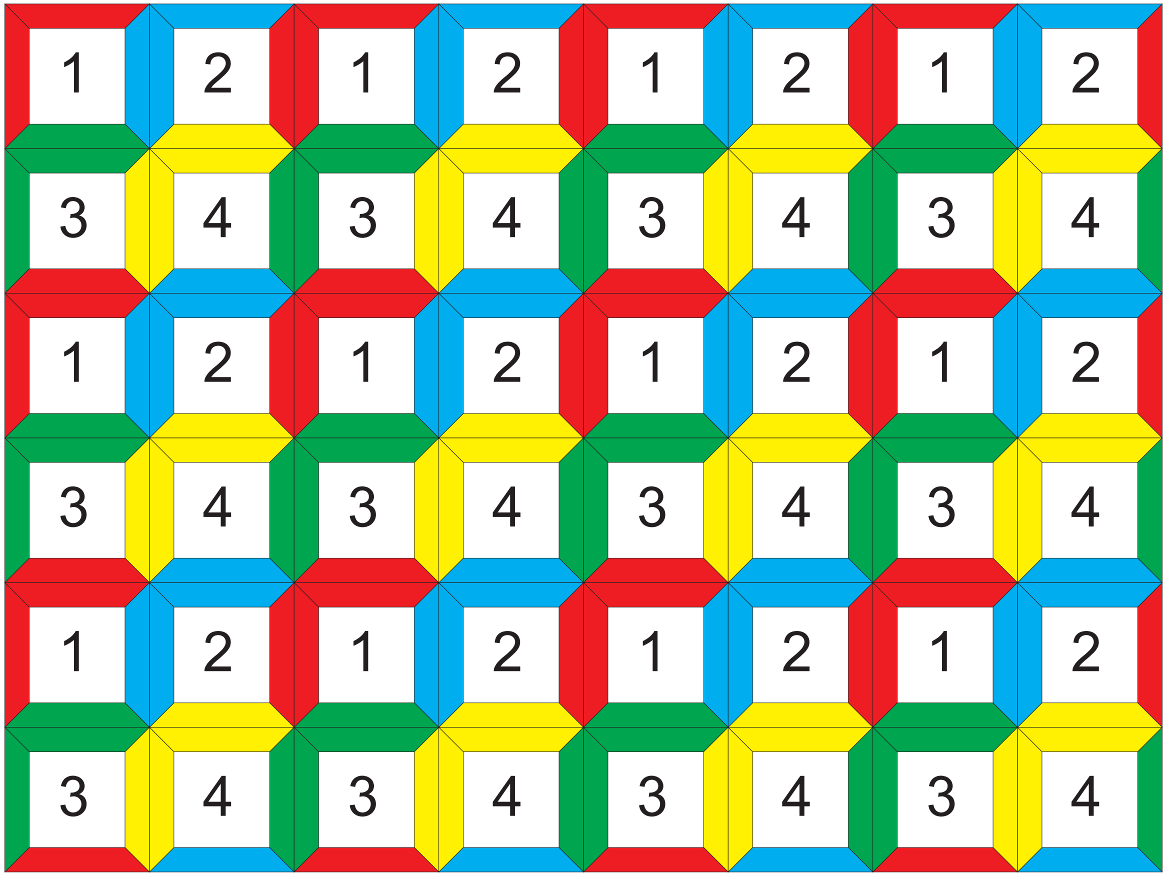
The tiles are paired with **parity tiles**



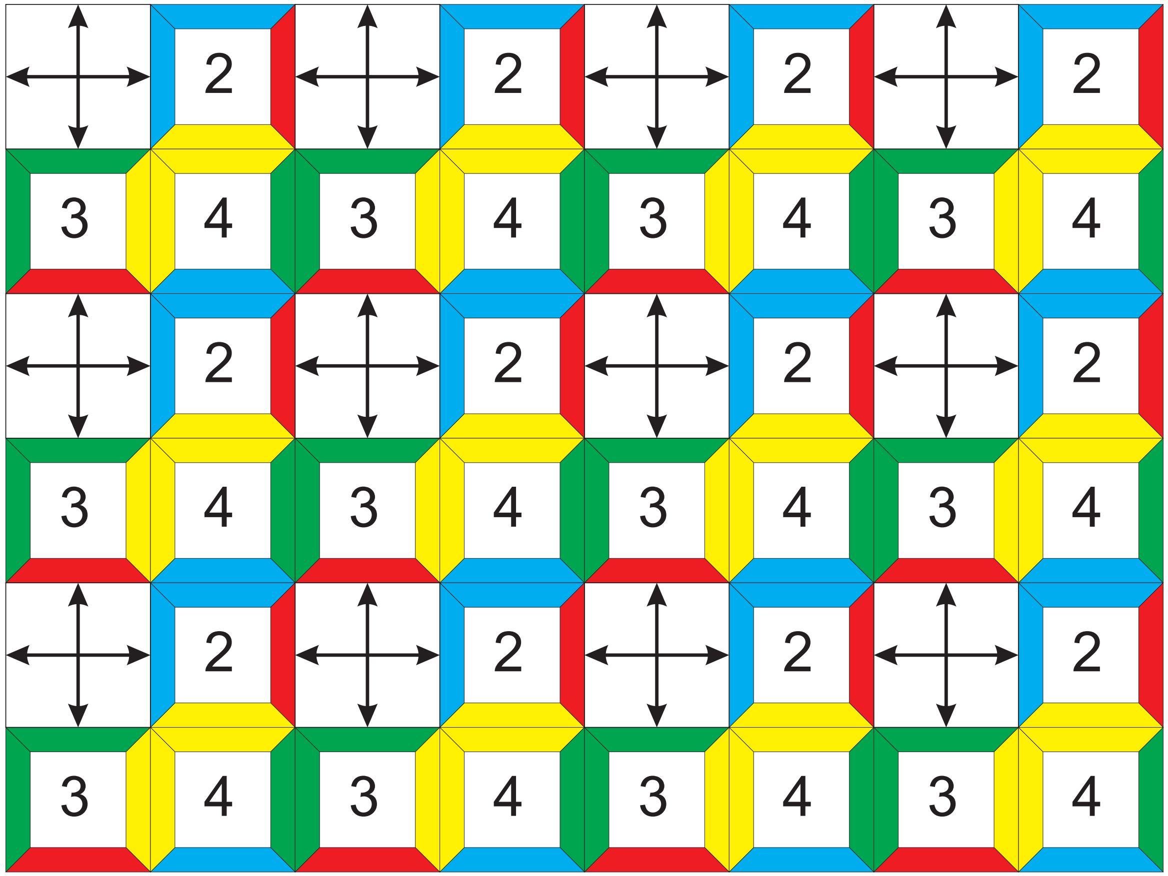
in such a way that

- **1** is only paired with crosses,
- **2** is only paired with vertically oriented arms,
- **3** is only paired with horizontally oriented arms.
- **4** can be paired with anything.

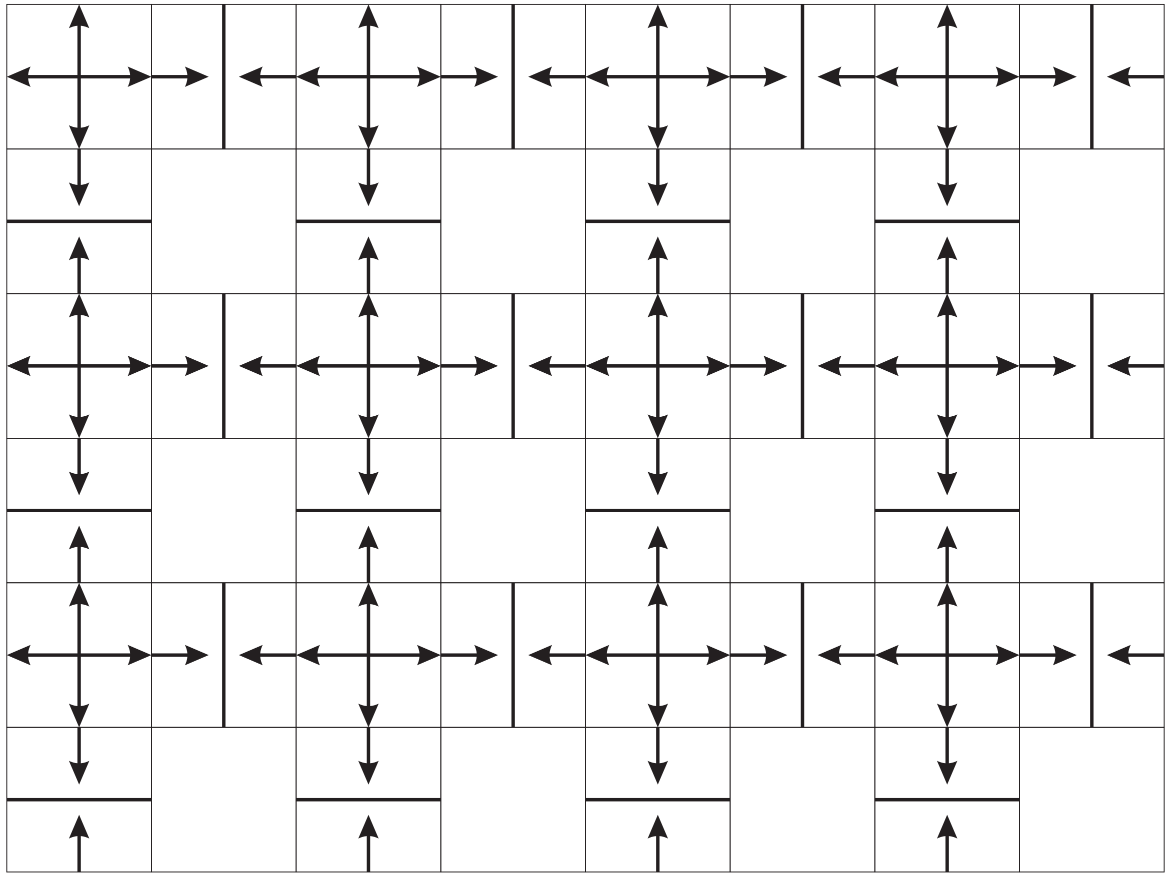
Each Robinson tile has then two possible parities (4 and either 1,2 or 3), so the total tile count is 56.



Tiling forced by parity tiles



Odd-odd positions (parity tile **1**) are forced to contain crosses.

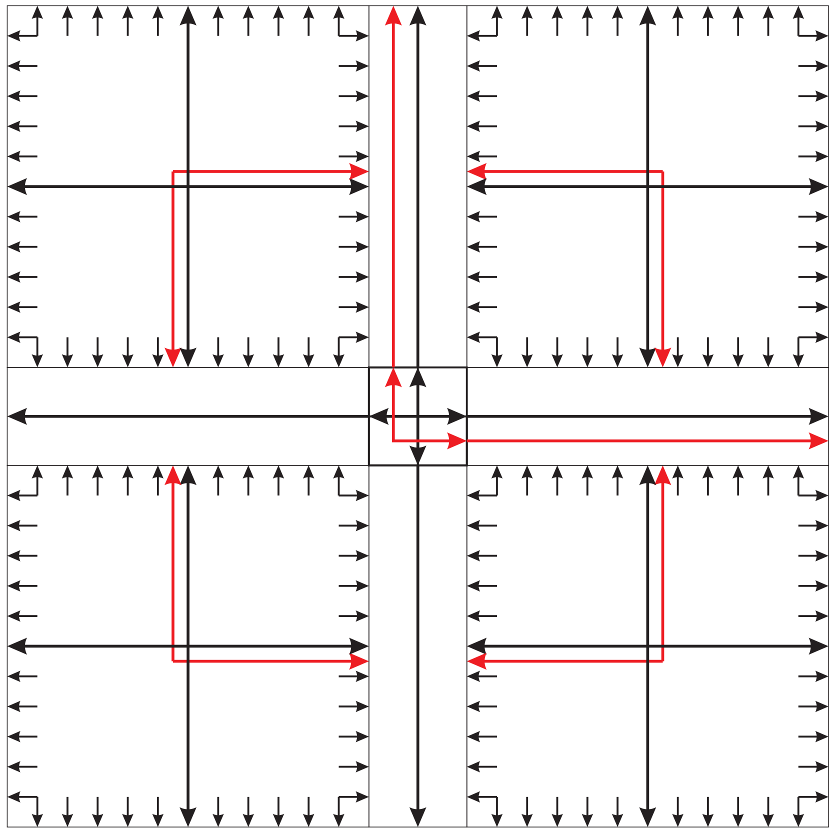


Odd-odd positions (parity tile **1**) are forced to contain crosses.

Vertical and horizontal arms are forced between them.

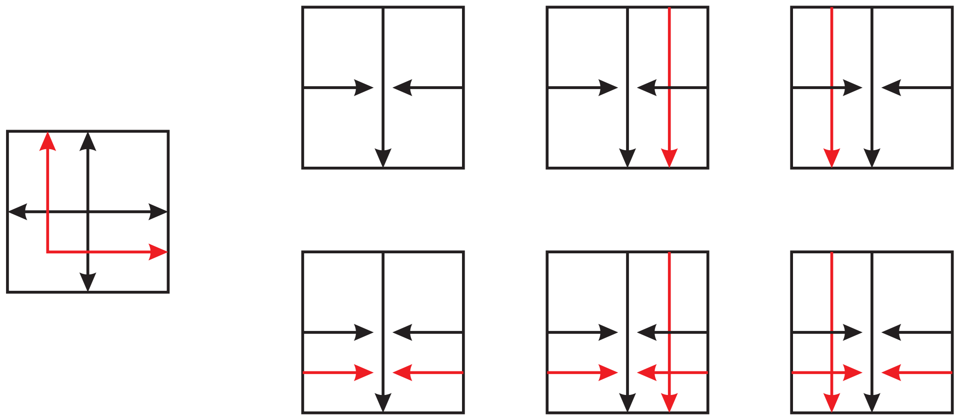
Tiles in even-even positions (parity tile **4**) can be chosen freely.

It was shown in **"Tilings and Patterns"** that Robinson's tiles do not admit a periodic tiling, but they admit tilings of arbitrarily large **special squares**

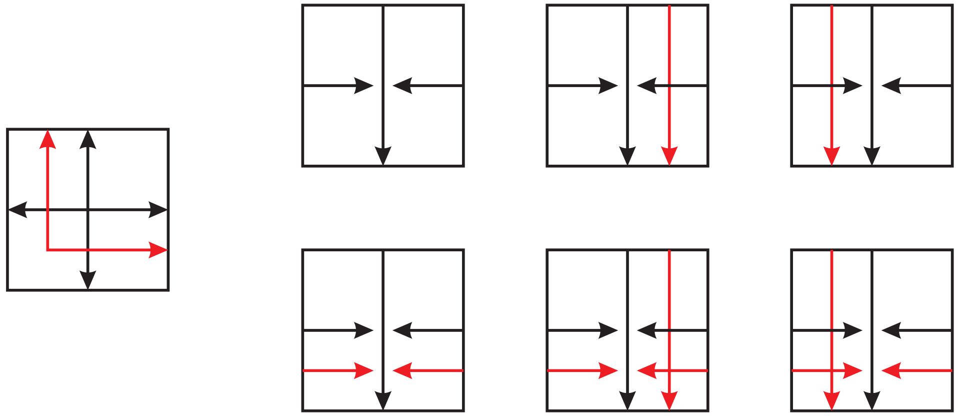


These are build iteratively by combining four smaller special squares into a larger one.

So the tile set admits a tiling and is **aperiodic**.

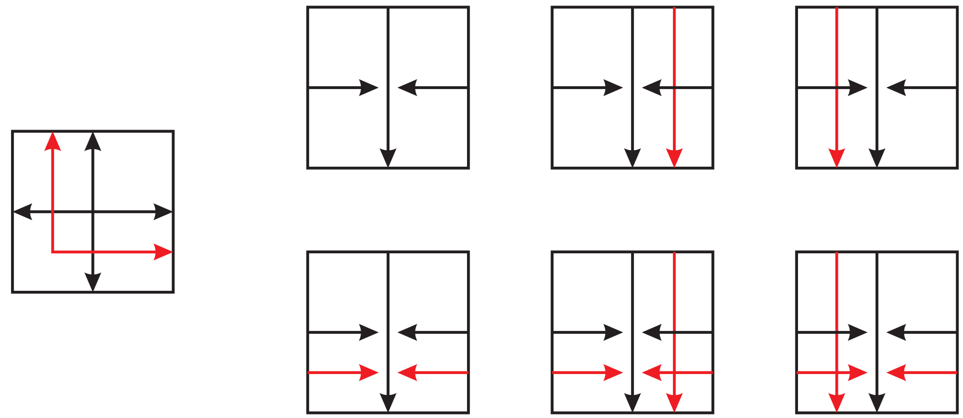


Robinson's tiles are **almost** NW-deterministic:



Robinson's tiles are **almost** NW-deterministic:

- One can NW-deterministically recognize whether a tile is a cross or an arm,



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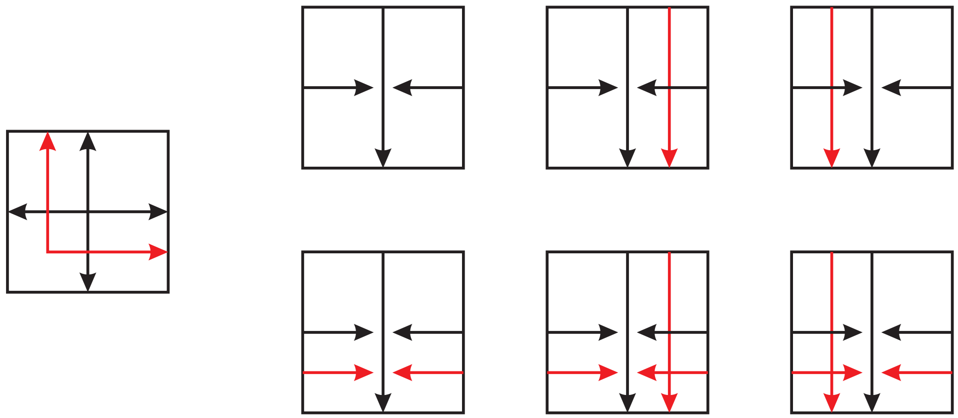
- The orientation of a cross is NW-deterministic,



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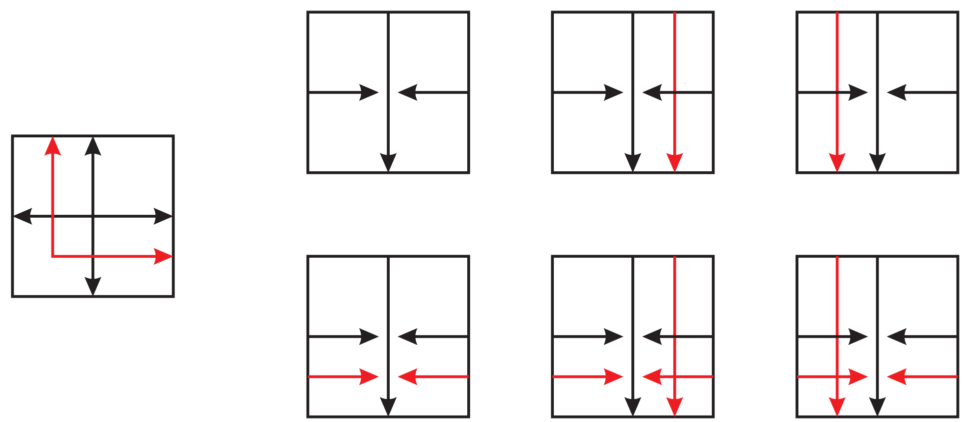
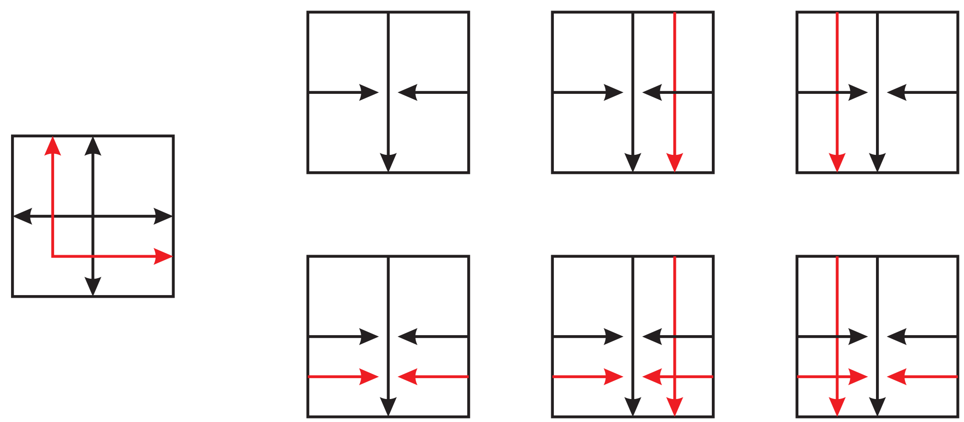
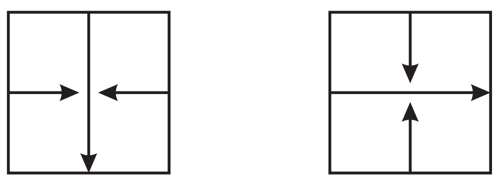


Robinson's tiles are **almost** NW-deterministic:

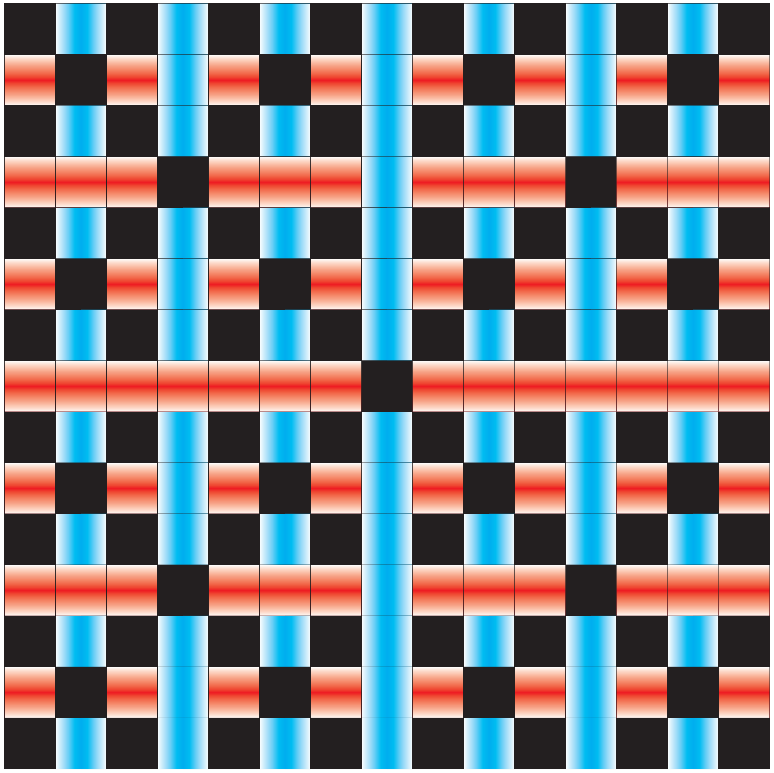
- One can NW-deterministically recognize whether a tile is a cross or an arm,
- The orientation of a cross is NW-deterministic,
- Parity-tiles are deterministic from all sides,
- Red side arrows are NW-deterministic, once the black arrows are known.



The only source of non-determinism is in identifying the direction of an arm, when both the north and the west sides have an incoming arrow:



Observation: In the special squares, horizontal and vertical arms alternate on each diagonal.



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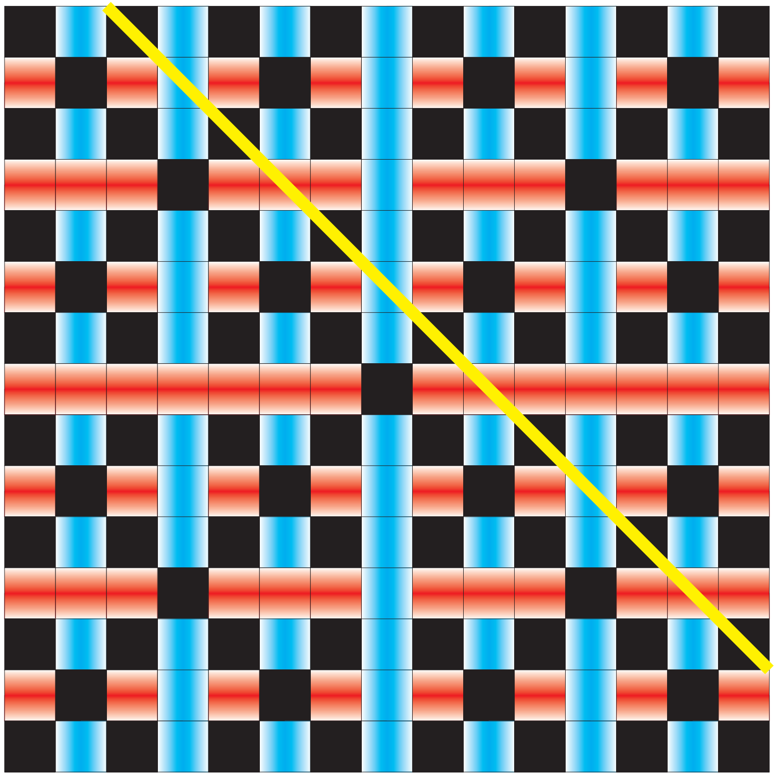

Hence, a layer of diagonal signals can identify whether an arm is horizontal or vertical.

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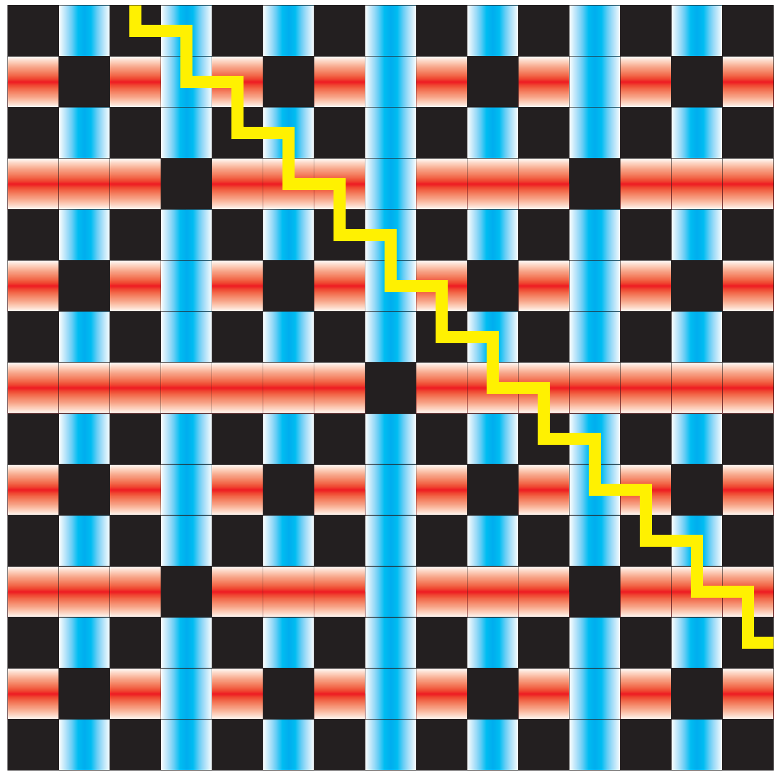

To use edge colors (instead of corner colors) the signal can be zigzagged.

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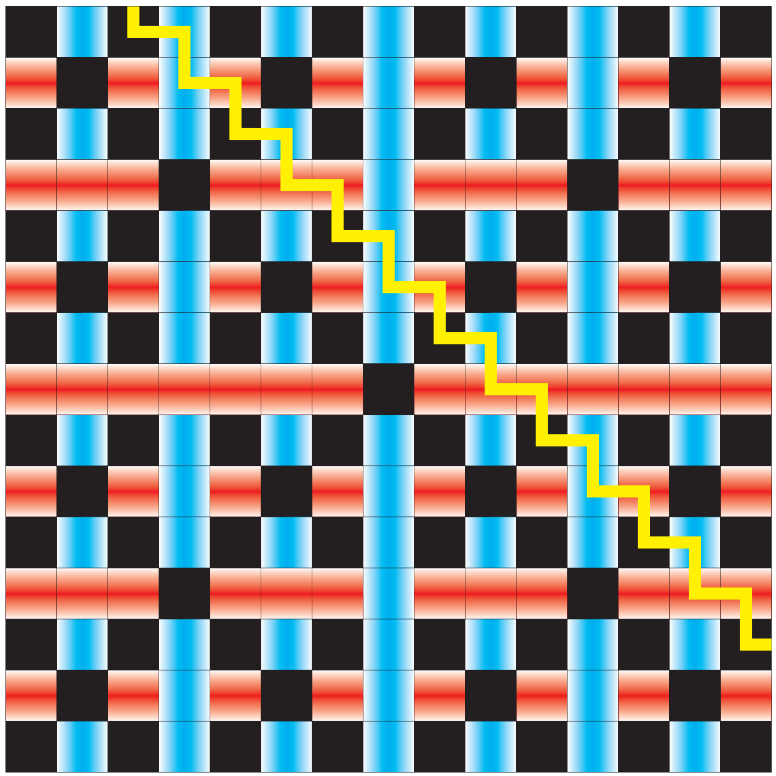
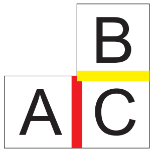


Thus we obtained a NW-deterministic, aperiodic Wang tile set.

Example. Let T be an aperiodic NW-deterministic tile set T. Let us construct a one-dimensional CA whose

- state set is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,
- neighborhood is $(0, 1)$,
- local rule $f : S^2 \longrightarrow S$ is defined as follows:
  - $f(A, B) = C$ if the colors match in



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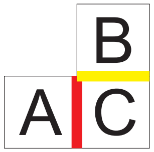

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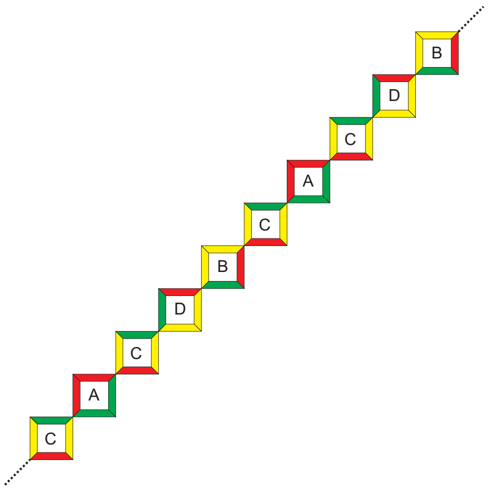

If configurations are drawn along diagonals, rule f puts matching tiles on the diagonal below.

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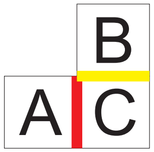

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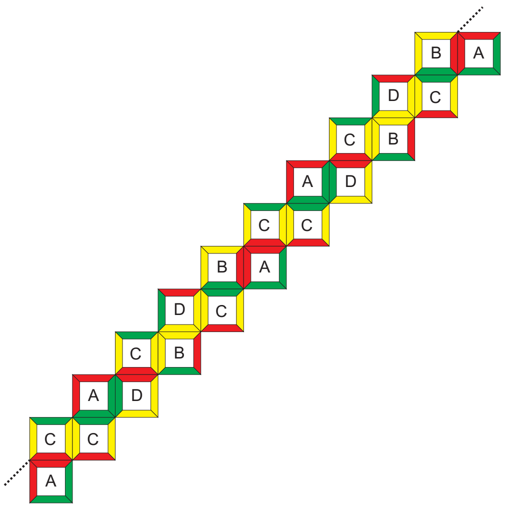

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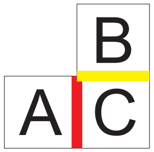

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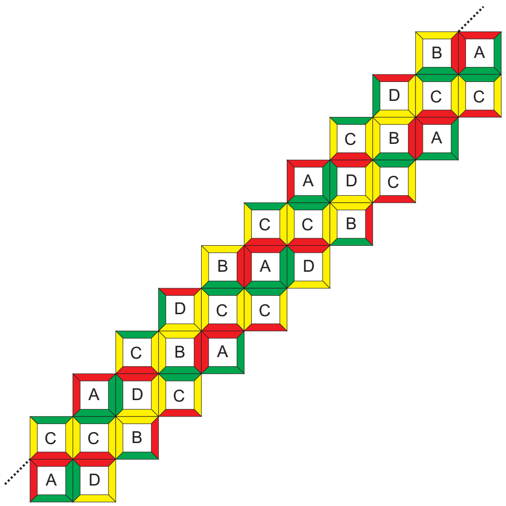

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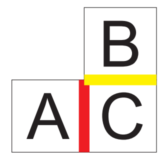

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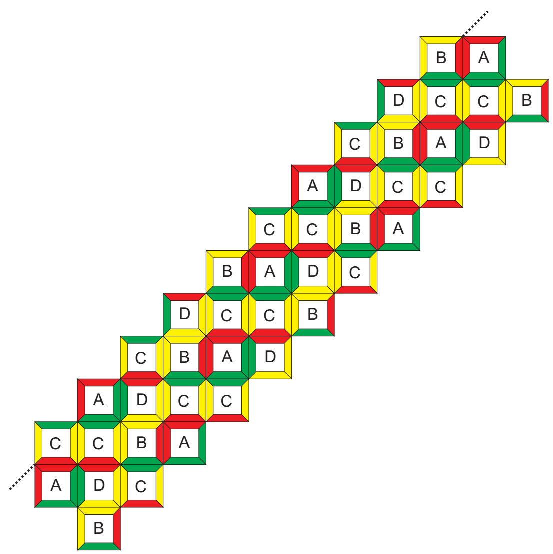

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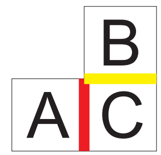

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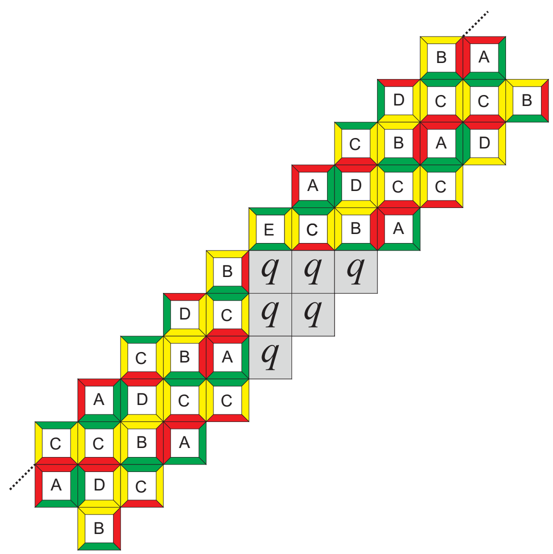

If no matching tile exists, a spreading "killer" state q is created.

Suppose a spatially **periodic initial configuration** c. Let us show that $G^n(c)$ becomes the q-unform configuration at some time n.

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But if the initial configuration is a diagonal of a valid tiling by T then $G^t(c)$ does not contain any cell in state q at any time t.

$\longrightarrow$ some spatially non-periodic configurations do not lead to any state q.

A **directed tile** is a tile that is associated a **follower vector** $\vec{f} \in \mathbb{Z}^2$.

Let $\mathcal{T} = (T, N, R)$ be a tile set, and let

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A sequence $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$ of cells is a (finite) **path** in t if

$$\forall i \qquad \vec{p}_{i+1} = \vec{p}_i + F(t(\vec{p}_i)).$$

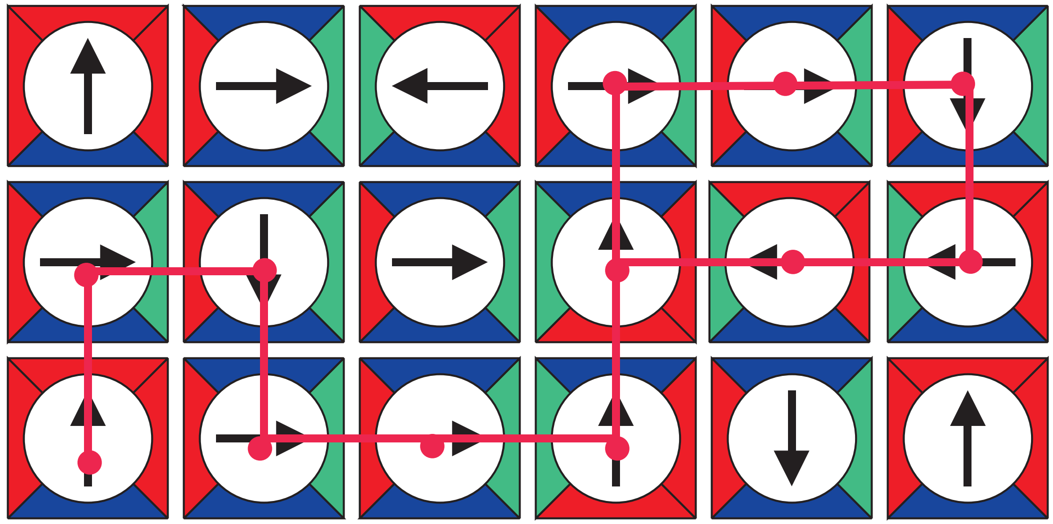
(A path is a sequence of cells where the next cell is always the follower of the previous cell.)

One-way infinite and **two-way infinite** paths are defined analogously.

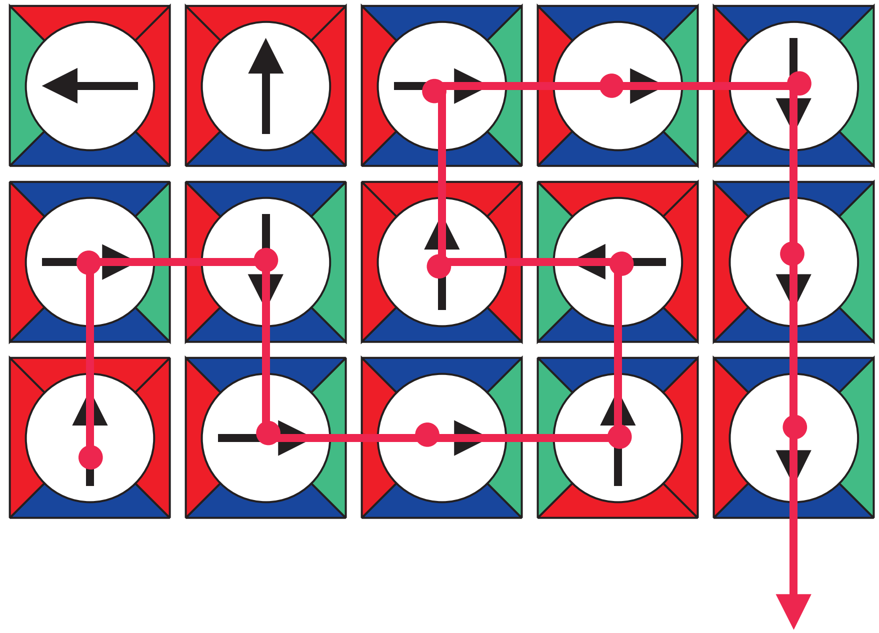
In the following the tiles are Wang tiles, and the followers are among the four adjacent positions:

$$F(a) \in \{(\pm 1, 0), (0, \pm 1)\} \text{ for all } a \in T.$$

The follower is drawn as an arrow on the tile:



Note that an infinite path may actually enter a loop so that it only covers finitely many cells.



Or it may cover infinitely many cells.

A set of directed tiles has the **plane-filling property** if

- it admits a **valid tiling** of the plane, and
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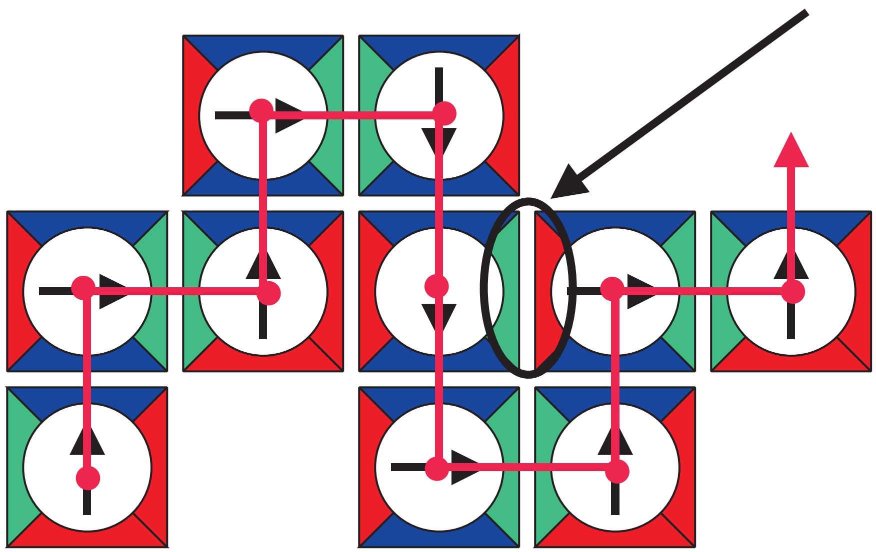

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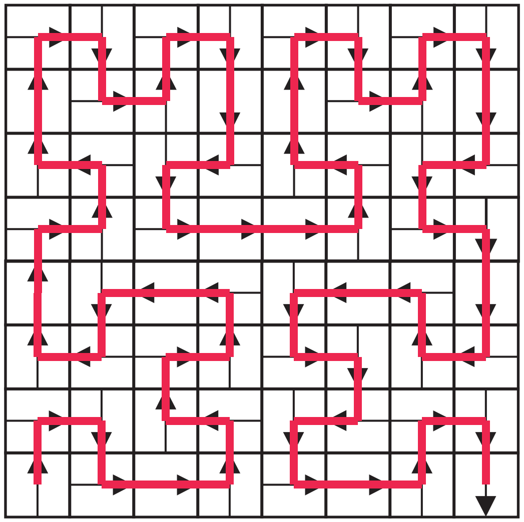

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Note that the tiling may be invalid outside path, yet the path is forced to snake through larger and larger squares.

In other words: the plane-filling property means that a **robot** that moves over a configuration t, verifies the validity of the tiling in its present location, and then moves on to the follower position, necessarily eventually either

- **finds a tiling error**, or
- **covers arbitrarily large squares**.

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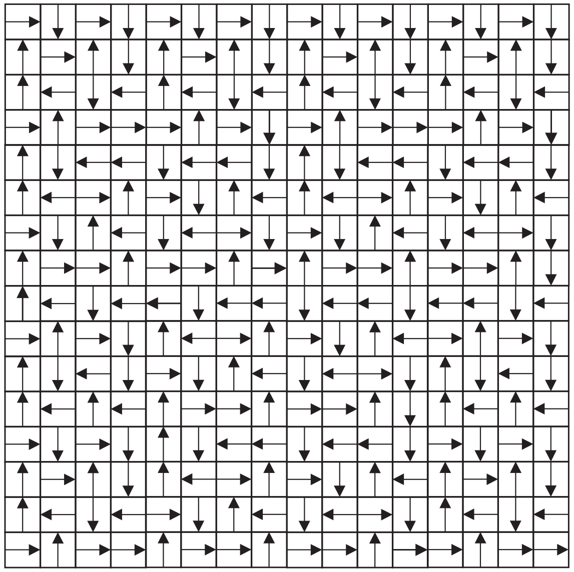
The paths that SNAKES forces when no tiling error is encountered have the shape of the well known plane-filling Hilbert-curve:



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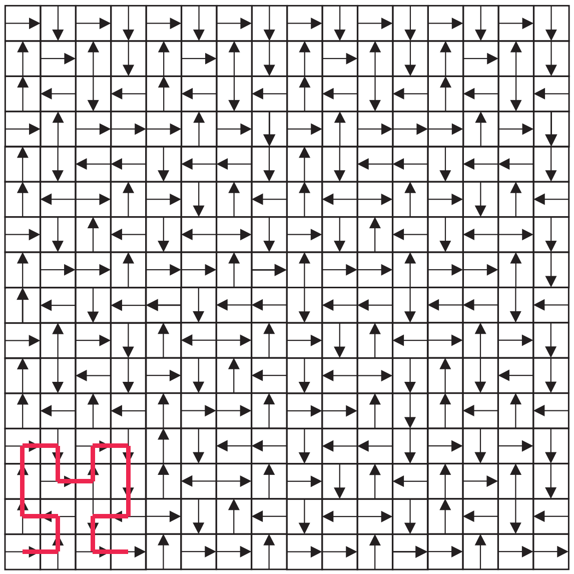

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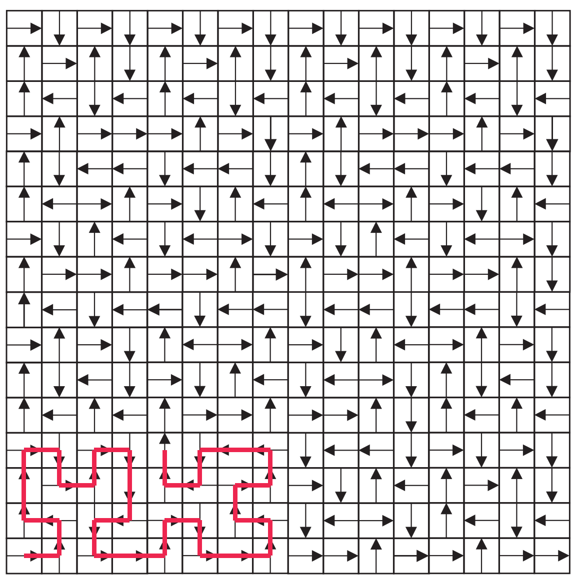

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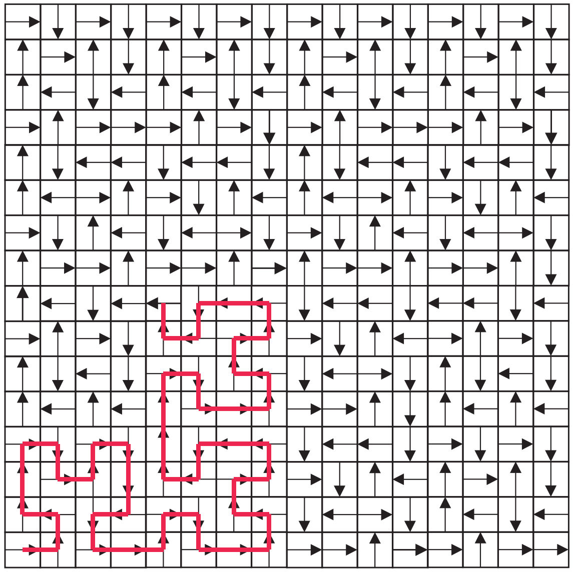

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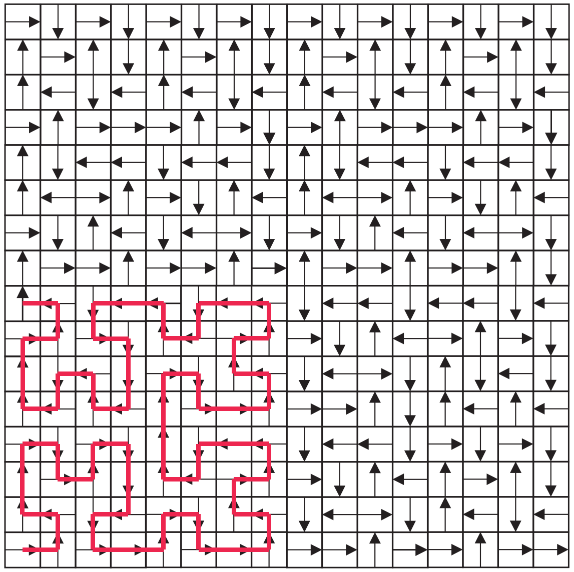

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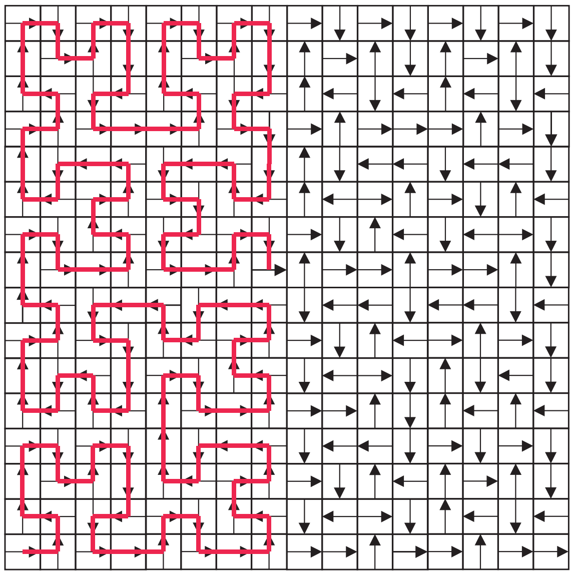

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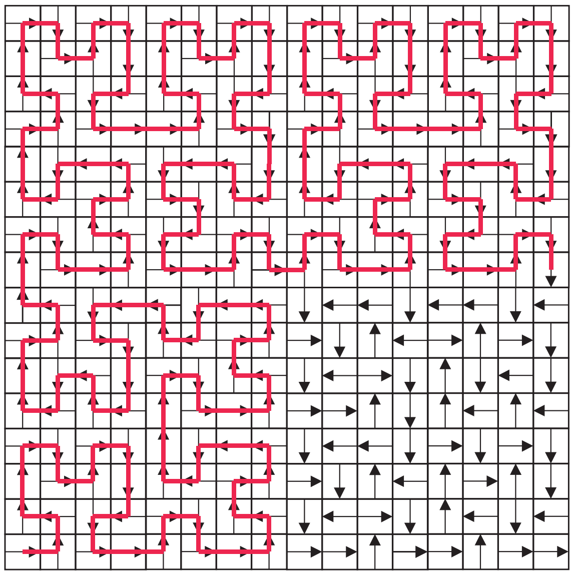

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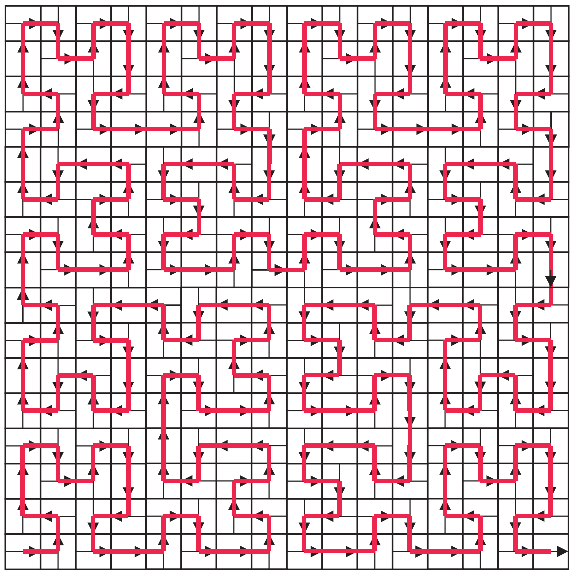
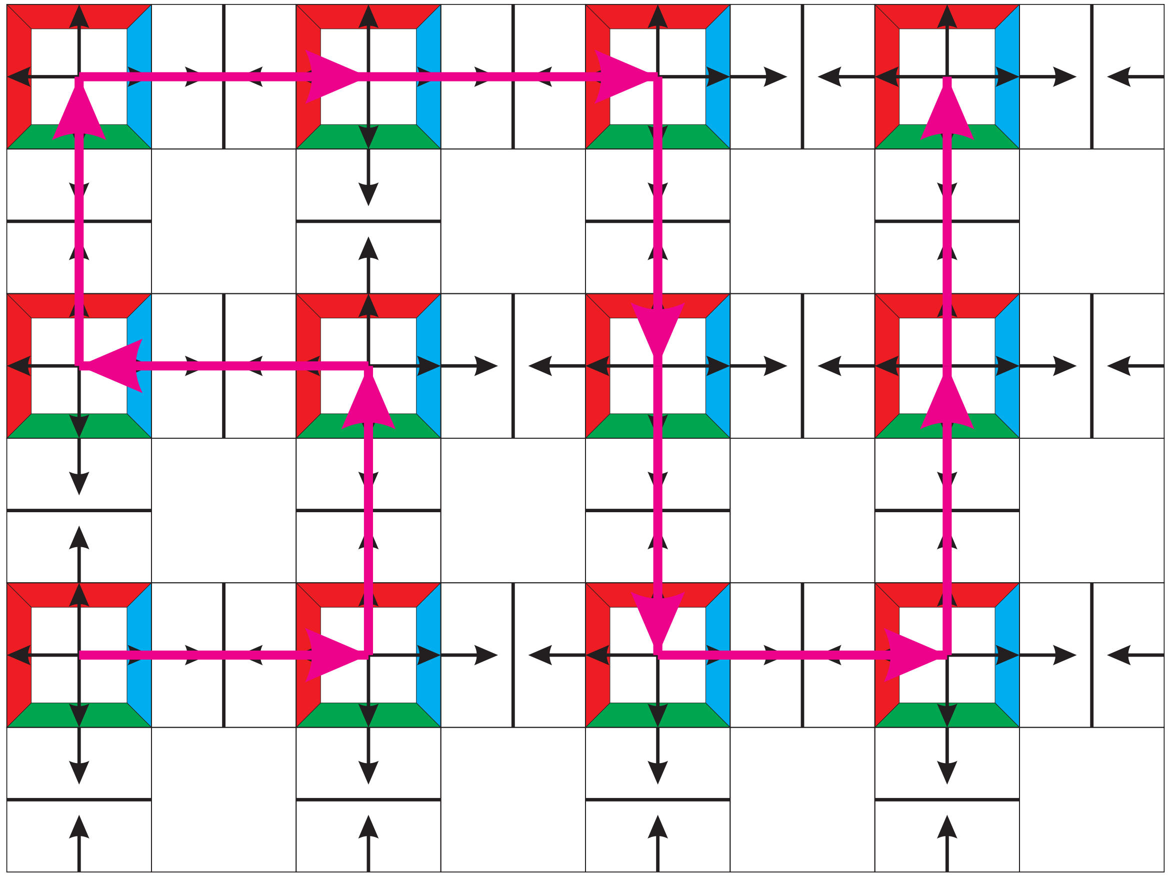


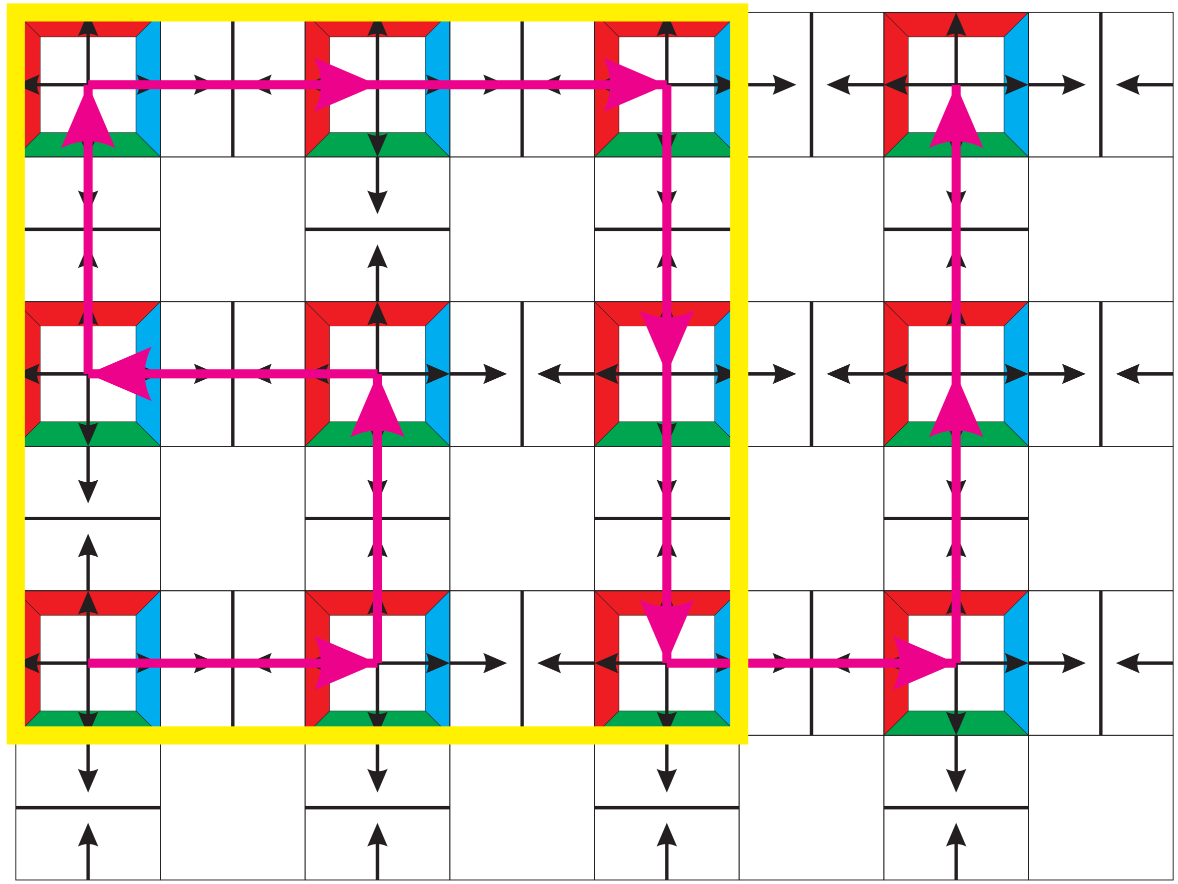
Robinson tiles $\Longrightarrow$ SNAKES

SNAKES are built using Robinson's tiles.

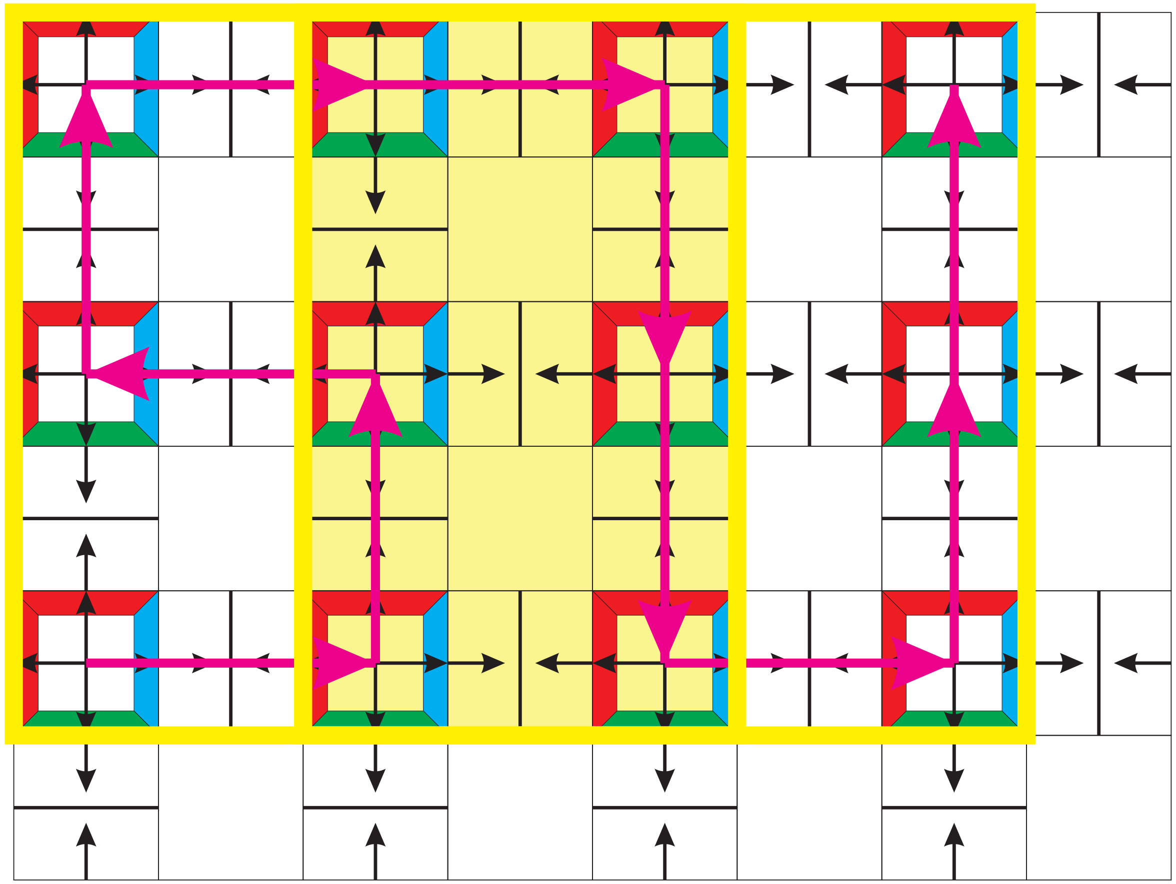


Directions are attached to crosses at odd-odd positions (parity tile 1), and each points to one of the four neighboring parity 1 crosses. Paths thus traverse through parity 1 crosses.

The Hilbert-curve is forced through special squares.

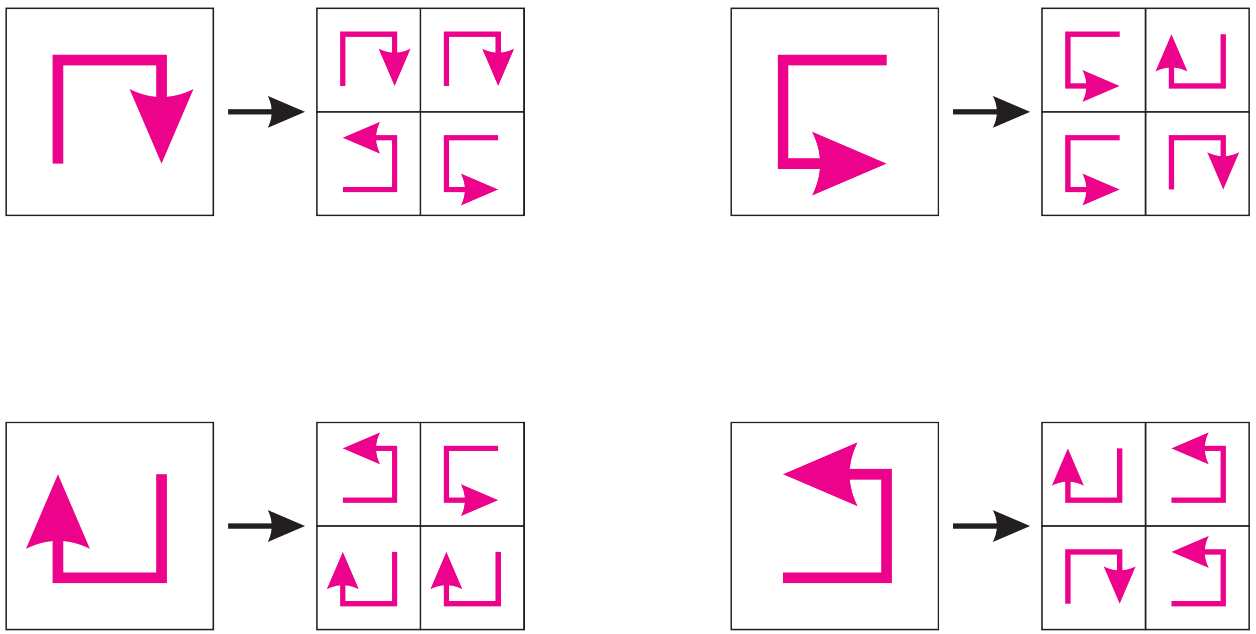


To allow the tiles to "see" further in the configuration, we take as tiles all 5×5 blocks that appear in correctly tiled special squares and that are centered at parity 1 tile.



The matching rule is that the overlapping regions (which are 3×5 or 5×3 subblocks) of neighboring tiles must be equal. This is called the **higher block presentation**.

Hilbert-curve comes in four orientations, generated by substitutions



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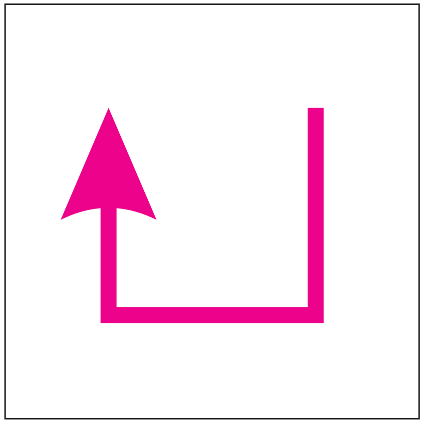

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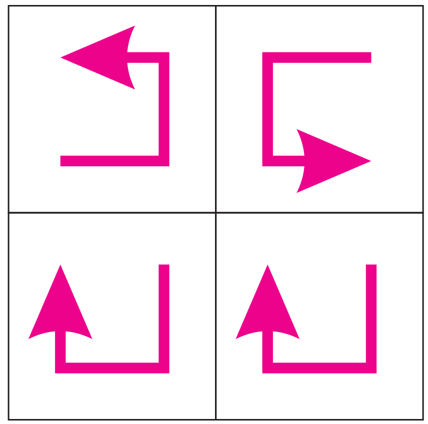

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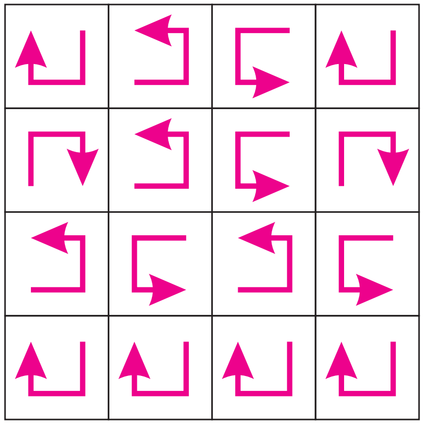
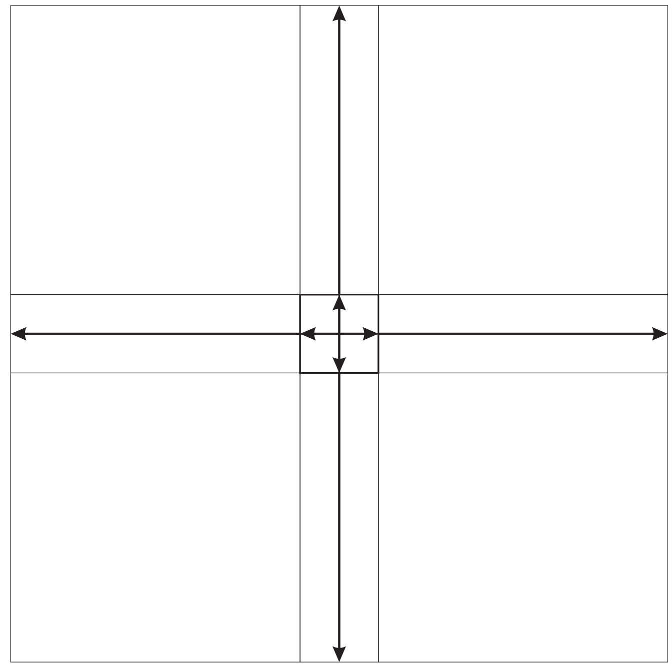
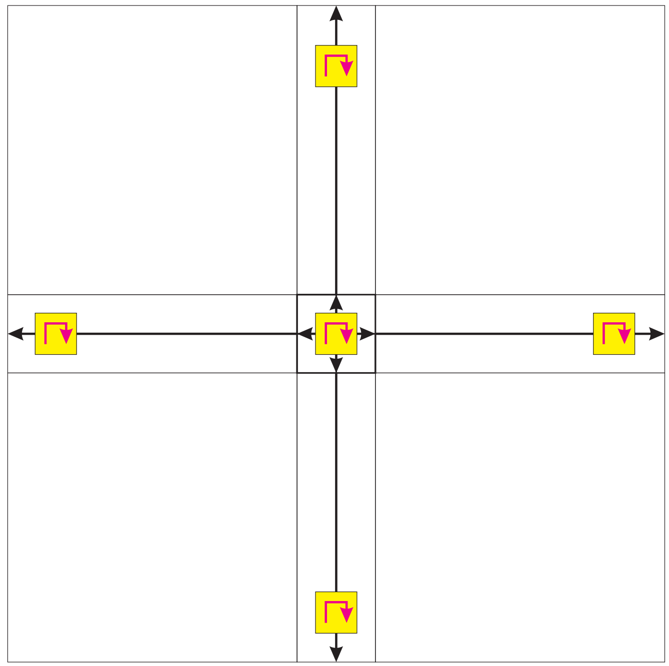
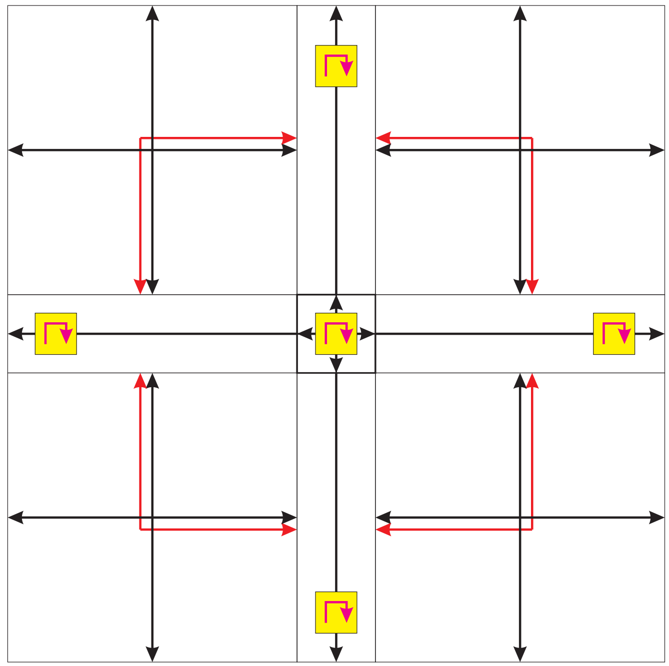




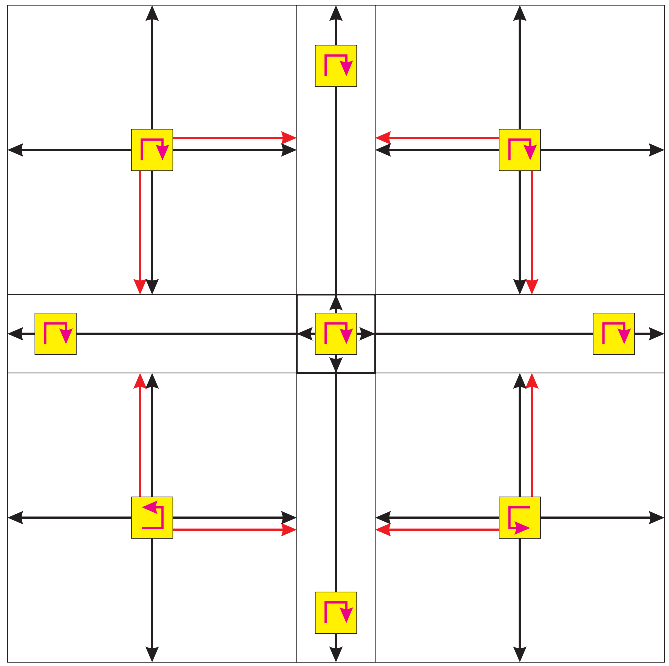
We want to force the Hilbert-curve through the special squares.



The center cross (and the arrows out of the cross) is labeled with a symbol that identifies the orientation of the curve through the square.



The four smaller squares at the corners should be labeled as given by the Hilbert substitution rule.



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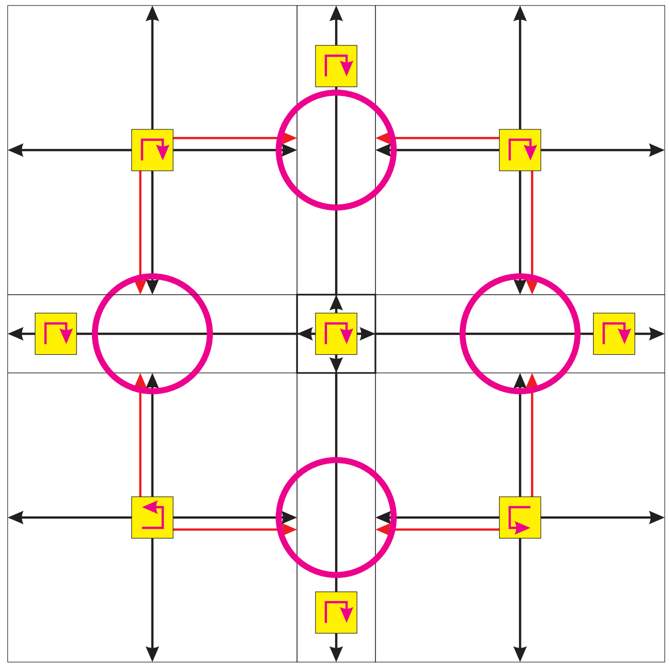
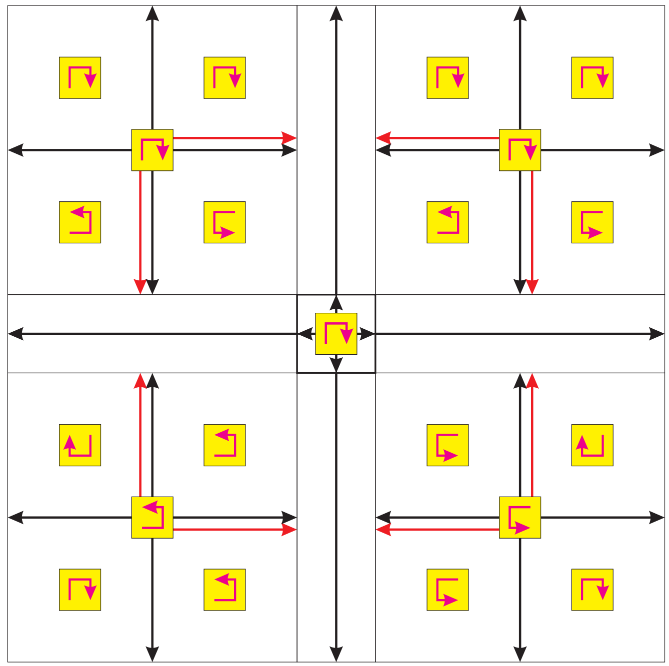


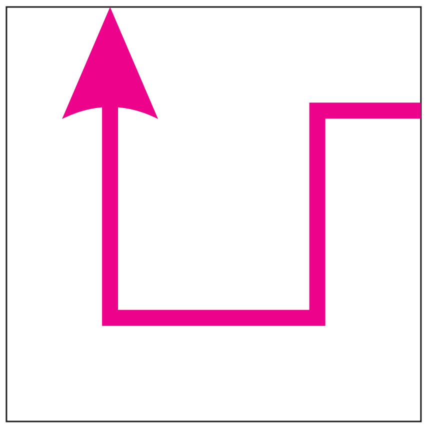
This is forced by limiting the allowed labels at the arms with incoming red side arrows.



This way the shapes are uniquely propagated to all crosses. The direction to be attached in each cross (in odd-odd position) can be uniquely deduced from these labels.

Our explanation was somewhat over-simplified:

It turns out that is more convenient to include in the labels the directions of entering and leaving the square, e.g.



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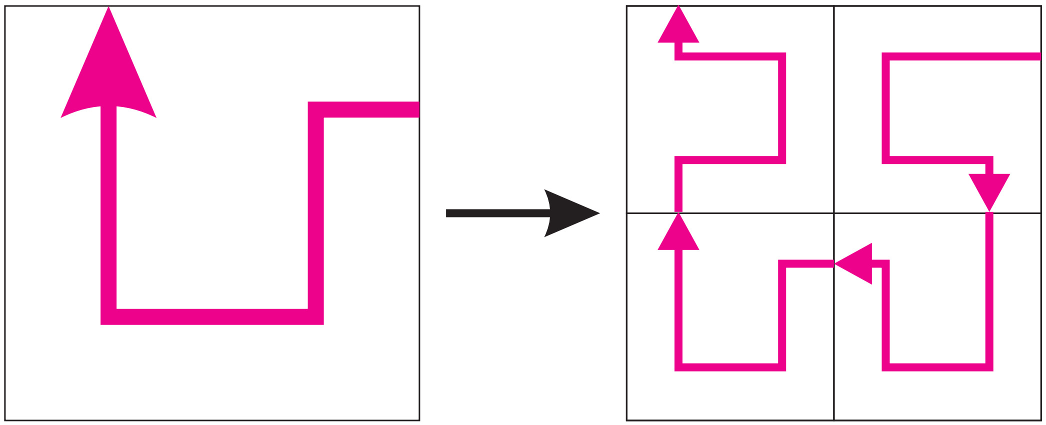

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Such construction (plus some more technical details) provides directed tile set SNAKES with the plane-filling property:

- Any infinite path where colors match between all neighboring tiles along the path is plane filling: it covers arbitrarily large squares.
- There exists a valid tiling of the plane. (In fact, a tiling exists where a single bi-infinite path covers all cells.)