Cellular Automata. Homework 7 (16.3.2026)

1. Determine if there exists a one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ for which the following decision problem is RE-complete: "Does a given spatially periodic initial configuration eventually become a configuration that contains some state from a fixed accepting state set $H \subseteq S$?"

2. A configuration $c \in S^{\mathbb{Z}}$ is called *recursive* if there exists an algorithm A that with input $n \in \mathbb{Z}$ returns $c(n)$. Let us call CA "universal" if the following question is RE-complete for some fixed accepting state $h \in S$: "Does a given recursive configuration c evolve into a configuration where cell 0 is in state h ?" (The recursive configuration is specified by giving an algorithm A that computes it.)

   Prove that the left shift σ is "universal" under this definition.

   [This means that this definition of "universality" is too general.]

3. Let G be cellular automaton. Let us call configuration c *strictly temporally periodic* (or STP) if it is temporally periodic but not spatially strongly periodic. Determine the STP configurations of the following one-dimensional cellular automata.

   (a) The identity function $i : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$.

   (b) The left shift $\sigma : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$.

   (c) The cartesian product $i \times \sigma$ of the identity and the left shift. (The product has the state set $\{0,1\} \times \{0,1\}$, and it applies i and σ on the first and the second components, respectively, independently of each other.)

4. Determine the STP configurations of the *xor* CA of Example 1 in the notes. (See Problem 3 above for the definition of STP.)

5. Let us call a one-dimensional CA $G : S^{Z} \longrightarrow S^{\mathbb{Z}}$ *positively expansive* if there is a positive integer n such that for any $c_1, c_2 \in S^{\mathbb{Z}}, c_1 \neq c_2$ there exists time $t \geq 0$ and cell i in the interval $-n \leq i \leq n$ such that $G^t(c_1)(i) \neq G^t(c_2)(i)$. In other words, if two configurations c_1 and c_2 differ somewhere then this difference will be eventually seen by an observer that sees only cells $-n, \dots n$.

   (a) Prove that a positively expansive cellular automaton has only a finite number of fixed points.

   (b) Prove that if G is positively expansive, so is G^k for all $k \geq 1$.

   (c) Prove that positively expansive CA do not have STP points (see Problem 3 for the definition of STP.).

6. Consider the two-dimensional CA G constructed in the proof of Proposition 45 for a given tile set T and a blank tile $B \in T$.

   (a) If the tile set T admits a valid, finite, non-trivial tiling of the plane, is G_P surjective ? (Recall that G_P is restriction of G on strongly periodic configurations.)

   (b) If the tile set T does not admit a valid, finite, non-trivial tiling of the plane, is G_P surjective ?

   (c) Is it decidable to determine for given two-dimensional CA G whether G_P is surjective ?

7. Langton's ant is the following two-dimensional CA: State set is

$$S = \{\text{Black}, \text{White}\} \times \{\leftarrow, \uparrow, \rightarrow, \downarrow, \perp\}.$$

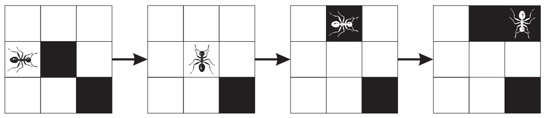
The arrow gives the direction of the ant positioned at the cell, and $\perp$ indicates that there is no ant at the cell. We define the local rule only in the case that neighborhoods contain at most one ant: the local rule consists then of the following two steps:

(1) move the ant one cell to its direction,

(2) change the direction of the ant 90° to the left or right, depending on whether the color of the cell is Black or White, respectively, and then swap the color

$$\text{White} \leftrightarrow \text{Black}$$

For example, here are three consecutive moves of an ant:



Prove that starting from any initial configuration containing a single ant, the ant will visit infinitely many different cells (i.e., the configuration is not eventually periodic).