Cellular Automata. Homework 8 (23.3.2026)

1. Let us call a CA *totalistic* if its state set is $\{1, 2, \ldots, s\}$ and the local rule f satisfies

$$s_1 + \cdots + s_m = r_1 + \cdots + r_m \Longrightarrow f(s_1, \ldots, s_m) = f(r_1, \ldots, r_m).$$

   (In other words: the sum of the states in the neighborhood determines the next state.) Which of the following CA are totalistic (after suitably renaming the states) ?

   (a) The *xor* CA of Example 1 from the notes.

   (b) The *majority rule* of the Homework set 1, problem 6.

   (c) The *traffic* CA defined in the Homework set 5.

2. Prove that a one-dimensional totalistic CA with at least two states and neighborhood vector $N = (1, 2, \ldots, m)$ of $m \geq 2$ consecutive cells is never reversible.

3. Construct a one-dimensional CA over the state set $\{0, 1, 2, 3, 4, 5\}$ that multiplies by 3 any positive integer expressed in base-6. More precisely, the rule should compute

$$G(\ldots 000 w_{n-1} \ldots w_0 000 \ldots) = \ldots 000 u_n u_{n-1} \ldots u_0 000 \ldots$$

   where

$$\sum_{i=0}^{n} 6^i u_i = 3 \left(\sum_{i=0}^{n-1} 6^i w_i \right).$$

   Then determine if your CA is reversible.

4. Using the CA of the previous problem, construct a one-dimensional CA over the state set $\{0, 1, 2, 3, 4, 5, \diamond\}$ that simulates the Collatz function $f : \mathbb{N} \longrightarrow \mathbb{N}$

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

   Positive integer n is represented by the configuration

$$c(n) = \ldots 000 w_{n-1} \ldots w_0 \diamond 000 \ldots$$

   where

$$n = \sum_{i=0}^{n-1} 6^i w_i.$$

   The CA should transform in one step any c_n into (a possibly shifted version of) $c_{f(n)}$.

In the last three problems we use the following concepts: A reversible CA G is called an *involution* if it is its own inverse, that is, if $G \circ G = id$. A reversible CA F is *time-symmetric* if there exists an involution G such that $F^{-1} = G \circ F \circ G$.

5. (a) Prove that F is time-symmetric if and only if it is the composition of two involutions.

   (b) Prove that if F is time-symmetric then also F^{-1} is time-symmetric, and F^n is time-symmetric, for all $n \in \mathbb{N}$.

   (c) Prove that if F is time-symmetric and H is reversible then $H \circ F \circ H^{-1}$ is time-symmetric.

6. Determine if the following 1D CA are time-symmetric:

   (a) The left shift σ,

   (b) The radius-0 CA with n states $1, 2, \ldots, n$ that rotates the states

   $$1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto n \mapsto 1.$$

   (c) The one-dimensional partitioned CA with state set $S = \{0,1\} \times \{0,1\} \times \{0,1\}$, neighborhood vector $N = (-1, 0, 1)$ and the local rule determined by the permutation $\pi : S \longrightarrow S$ that maps

   $$(a, b, c) \mapsto (a, a \oplus b \oplus c, c).$$

   Here, "$\oplus$" denotes the modulo two addition.

7. (a) Let $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ be any reversible CA with the state set S. Prove that $F = G \times G^{-1}$ is time-symmetric: The state set of F is $S \times S$. In the first component, CA G is executed, while on the second component, CA G^{-1} is executed.

   [More precisely: F maps $(c_1, c_2) \mapsto (G(c_1), G^{-1}(c_2))$, for any $c_1, c_2 \in S^{\mathbb{Z}^d}$, where we denote by (c_1, c_3) the element of $(S \times S)^{\mathbb{Z}^d}$ whose first and second layers read c_1 and c_2, respectively, that is, for all cells $\boldsymbol{n} \in \mathbb{Z}^d$: $(c_1, c_2)(\boldsymbol{n}) = (c_1(\boldsymbol{n}), c_2(\boldsymbol{n}))$.]

   (b) Prove that there exists a one-dimensional time-symmetric CA that is universal in the sense that the decision problem: "Does a given finite initial configuration evolve into a configuration in which the state of some cell belongs to A ?" is RE-complete for some fixed $A \subseteq S$.