Cellular Automata. Homework 9 (30.3.2026)

1. Recall (last week homework set, Problem 3) the one-dimensional CA F that multiplies by constant 3 numbers expressed in base 6. It has the state set $S = \{0, 1, 2, 3, 4, 5\}$, radius-$\frac{1}{2}$ neighborhood $N = (0, 1)$ and the local rule $f : S^2 \longrightarrow S$ that maps for all $x_1, x_2 \in \{0, 1, 2\}$ and $y_1, y_2 \in \{0, 1\}$
$$f(2x_1 + y_1, 2x_2 + y_2) = 3y_1 + x_2.$$
Show that F is a partitioned CA (and hence reversible) after renaming the state set S as a cartesian product $S_1 \times S_2$ of suitable sets S_1 and S_2.

2. Recall the one-dimensional reversible CA of Example 4, page 19 in the notes. It has three states, neighborhood $N = (0, 1)$ and the local rule $f(a, b)$ given by the table

a \ b	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

   (a) Find all left and right stairs when radius $r = 1$ is used.
   (b) Show that the CA can be defined using one-dimensional Margolus neighborhood, that is, as a composition of two permutations on overlapping partitionings of the configuration into segments of length two.
   (c) Show that the 2-block presentation of $G \circ \sigma$ is partitioned (when the states are renamed appropriately). Give the partitioning $S_1 \times S_2$ used, and the permutation π.

3. Let us generalize the CA of Problem 2 to n-states, $n > 3$: Consider the CA that has the radius-$\frac{1}{2}$ neighborhood $N = (0, 1)$, states $1, 2, \ldots n$ and the local transition rule given by
$$f(a,b) = \begin{cases} a, & \text{if } a \geq b, \\ 1, & \text{if } a = b - 1, \\ a+1, & \text{if } a < b - 1. \end{cases}$$
Prove that the CA is reversible and determine the minimum neighborhood of its inverse CA.

4. (a) Show that it is undecidable if a given two-dimensional CA G that is an involution (i.e., $G^{-1} = G$) has any fixed points.
   (b) Show that it is undecidable if a given partitioned two-dimensional CA G with the von Neumann neighborhood has any fixed points.

5. Determine how the HPP lattice gas (defined in Section 4.2 of the notes) evolves from a finite initial configuration. More precisely, show that if the particles initially fit inside a $w \times h$ box then after $\max\{w, h\}$ steps no more particle collisions happen, and all particles continue on straight trajectories.

6. Prove that there is an algorithm to determine for given finite configurations c_1 and c_2 whether c_1 evolves into c_2 under the BBM cellular automaton (defined in Section 4.2 of the notes). (So BBM is not "computationally universal" on finite configurations.)

7. Prove that there is a one-dimensional totalistic CA that is computationally universal. (Recall the definition of totalistic CAs from last week exercises.) More precisely, show that there is a totalistic CA and fixed subset F of states for which the following decision problem is r.e. complete: "Does a given finite initial configuration evolve into a configuration in which the state of some cell is in F ?"