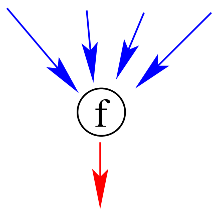
Next: All one-dimensional RCA are PCA (after suitably merging the cells).

Reversibility in CA is defined as a **global** property: the transformation

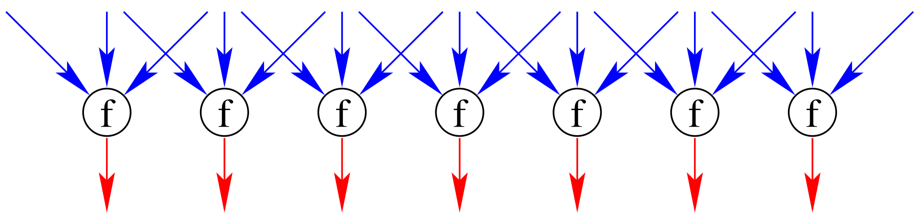
$$G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$$

is bijective on infinite configurations.

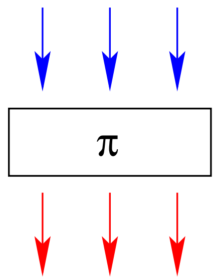
The **local** rules $f : S^n \longrightarrow S$ themselves are **not** reversible as the domain is larger than than the range:



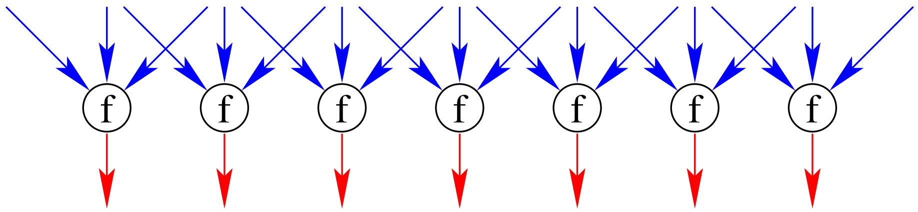
The non-reversible local rules "collaborate" to define a bijective transformation:



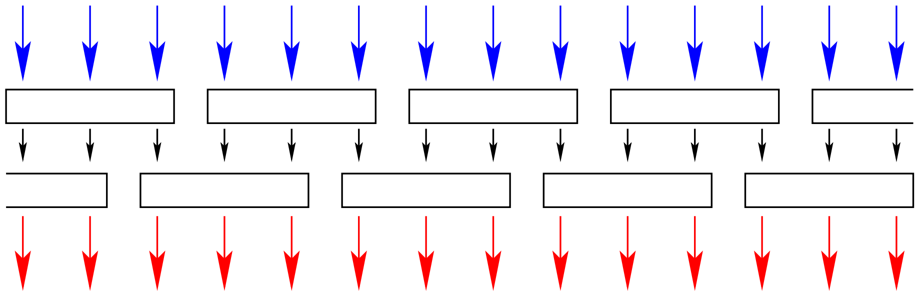
We would like to build the same transformation using reversible local rules, that can be then implemented using reversible logic gates:



In other words, we would like to replace a traditional CA



with something like

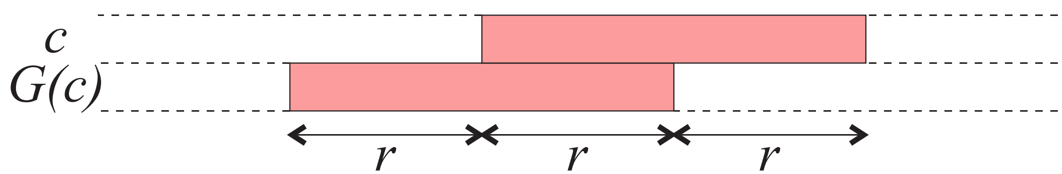


that computes the same global function $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$.

Let $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be any one-dimensional reversible cellular automaton.

Let r be a positive number such that both G and G^{-1} can be defined using the radius-r neighborhood.

A **right stair** is a pair of patterns of length $2r$ extracted from configurations c and $G(c)$ in a staggered way:

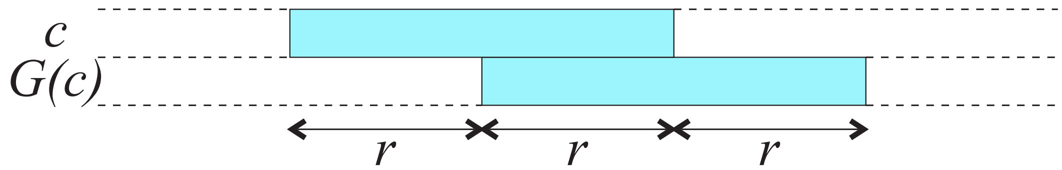


Precisely, the set of right stairs is

$$R = \{(c_{[0,2r-1]}, G(c)_{[-r,r-1]}) \mid c \in S^{\mathbb{Z}}\} \subseteq S^{2r} \times S^{2r}.$$

(We denote $x_{[n,m]}$ for $x(n)x(n+1)\dots x(m)$.)

Analogously, left stairs are extracted as follows:

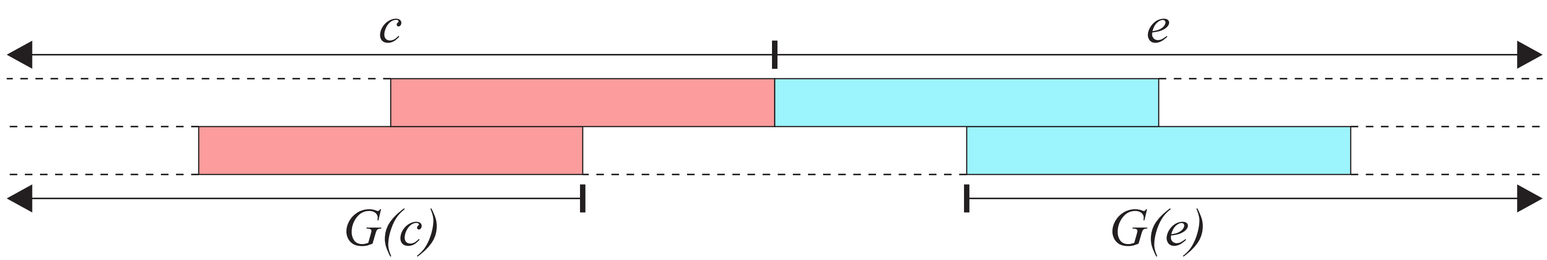


The set of left stairs is

$$L = \{(G(c)_{[0,2r-1]}, c_{[-r,r-1]}) \mid c \in S^{\mathbb{Z}}\} \subseteq S^{2r} \times S^{2r}.$$

(Note that L of G is exactly the same set as R of G^{-1}.)

Any right stair (extracted from some $c \in S^{\mathbb{Z}}$) and any left stair (extracted from some $e \in S^{\mathbb{Z}}$) can be extracted from the same configuration back-to-back:



(The three parts of stairs are at least as long as the radius of the CA.)

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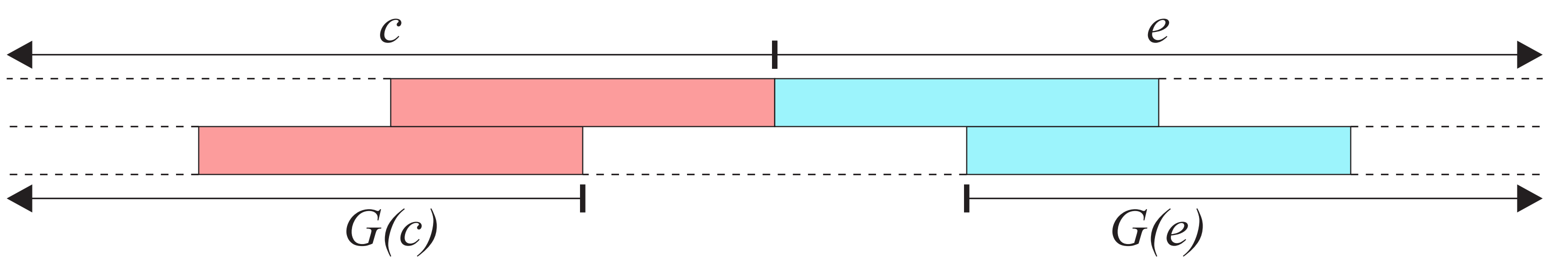
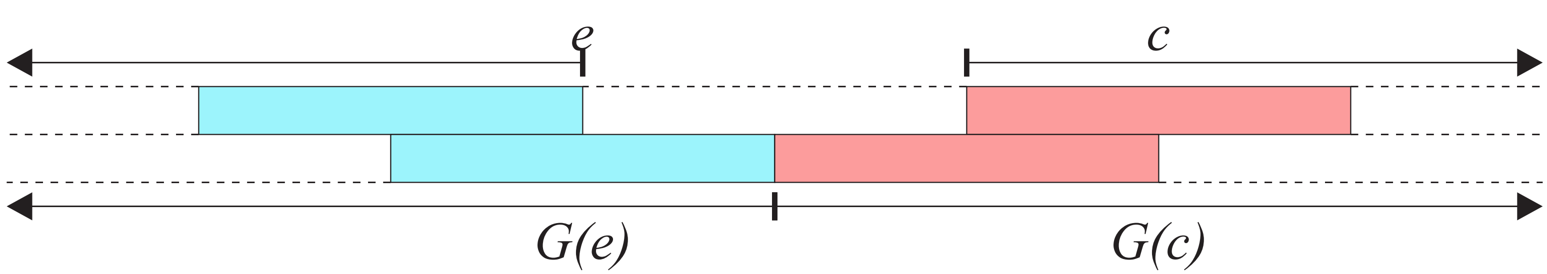


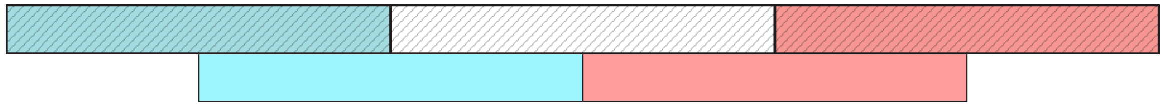
(The three parts of stairs are at least as long as the radius of the CA.)

Analogously, using the local rule of G^{-1}, the stairs can also be extracted from the same configuration back-to-back in the reverse order:



(The three parts of stairs are at least as long as the radius of the inverse CA.)

Using the local rule of G, any segment of length $6r$ (striped in the figure) uniquely determines a pair of left and right stairs:



Let us denote this function by

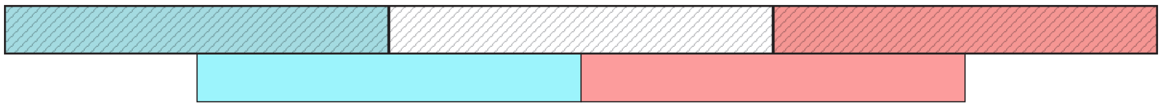
$$\varphi : S^{6r} \longrightarrow R \times L.$$

It maps for every $c \in S^{\mathbb{Z}}$

$$c_0 c_1 \ldots c_{6r-1} \mapsto \left[(c_{[4r,6r-1]}, G(c)_{[3r,5r-1]}), (G(c)_{[r,3r-1]}, c_{[0,2r-1]}) \right].$$

(The result is independent of the values of c outside the segment $\{0, \ldots, 6r-1\}$.)

$$\varphi : S^{6r} \longrightarrow R \times L$$



The function φ is

(a) **well defined**: The states in c that are outside the striped segment have no influence on the two stairs.

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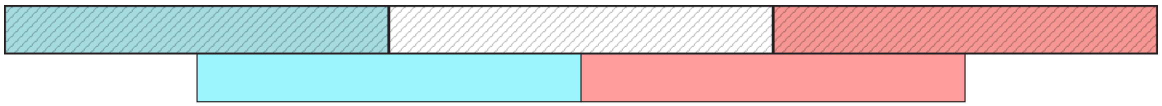

The function φ is

(a) **well defined**: The states in c that are outside the striped segment have no influence on the two stairs.

(b) **one-to-one**: Using the local rule of G^{-1} the lower halves of the two stairs uniquely determine the middle part of the striped segment.

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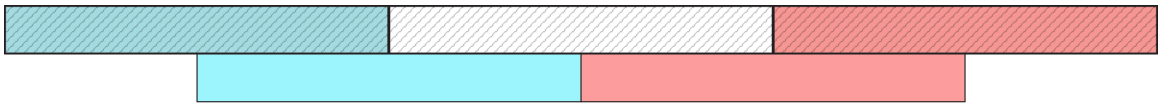

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(c) **surjective**: Any pair of left and right stairs can be extracted back-to-back from the same configuration, so for any pair of stairs a striped segment exists that is mapped by φ to the pair of stairs.

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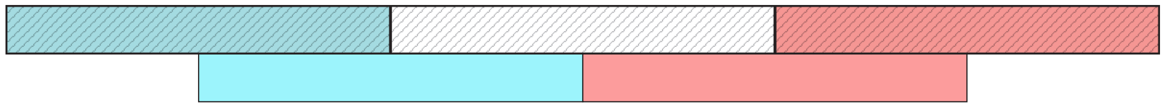

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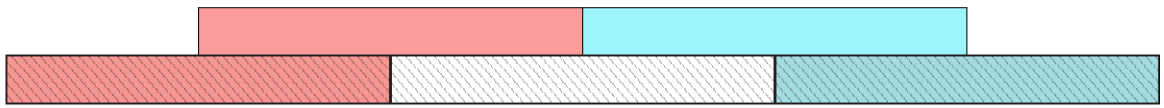
(c) **surjective**: Any pair of left and right stairs can be extracted back-to-back from the same configuration, so for any pair of stairs a striped segment exists that is mapped by φ to the pair of stairs.

Function φ is a **bijection**. (And, in particular, $|L| \cdot |R| = |S|^{6r}$.)

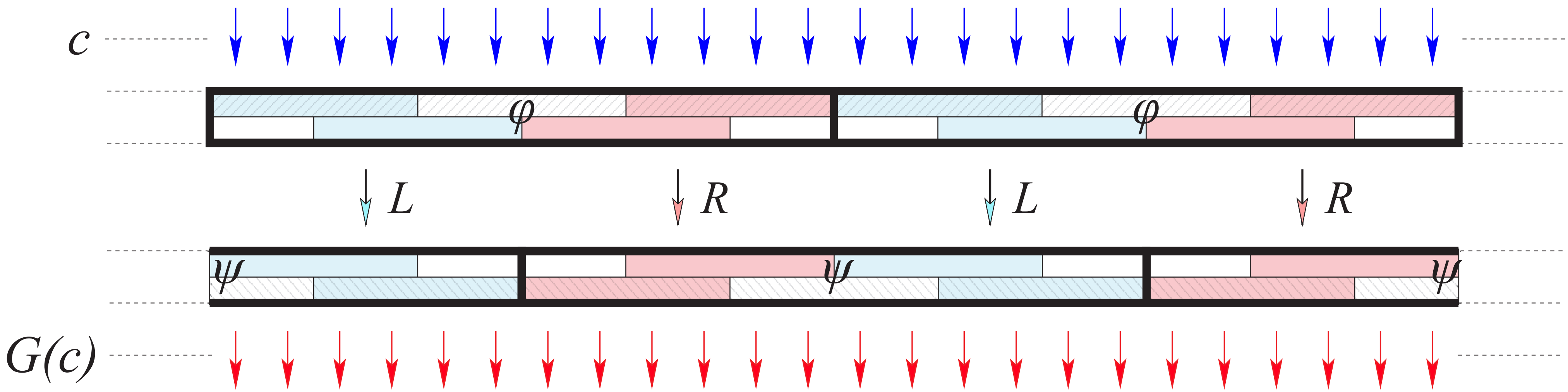
Analogously, we have a bijection

$$\psi : R \times L \longrightarrow S^{6r}$$

that maps a right stair/left stair pair into the $6r$ cells wide segment of $G(c)$, shown striped:



Now we can implement G using local bijections φ and ψ:



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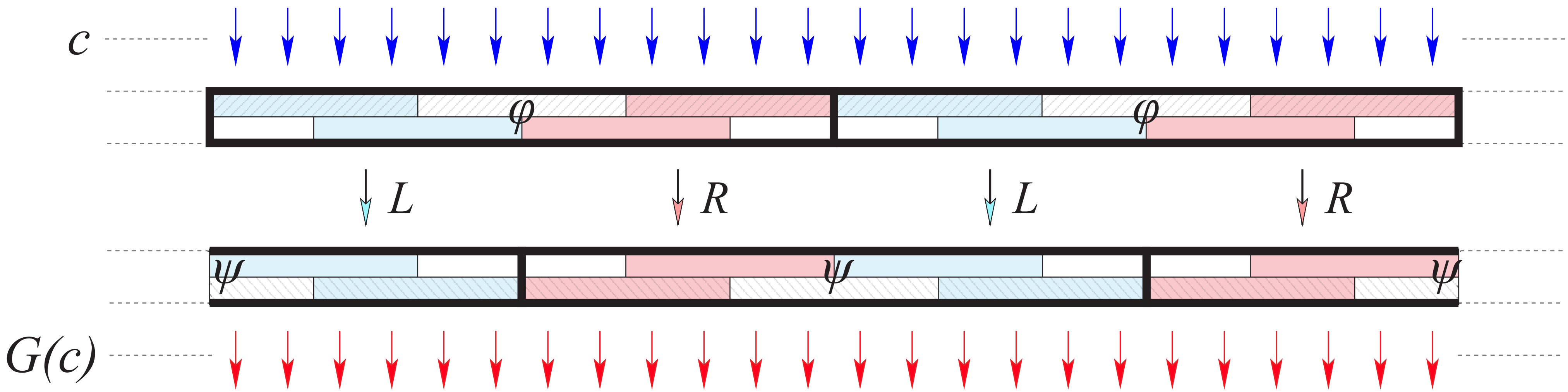


Let us see next how this construction shows that all one-dimensional RCA are PCA, after suitable merging of cells.

Notation 1:

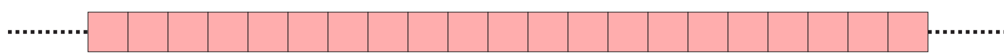
Let $m \in \mathbb{Z}_+$, and let S be the finite state set. We define a **blocking function**

$$B_m : S^{\mathbb{Z}} \longrightarrow (S^m)^{\mathbb{Z}}$$

that merges segments of m consecutive cells into "super cells": For every $c \in S^{\mathbb{Z}}$ we have $B_m(c) = e$ where for every $i \in \mathbb{Z}$

$$e(i) = (c_{mi+1}, c_{mi+2}, \ldots, c_{mi+m}) \in S^m.$$

For example B_3:



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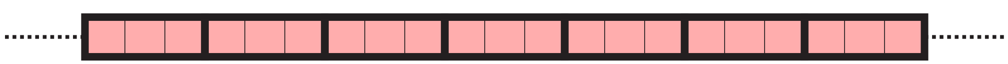

Clearly B_m is a bijection so it has the inverse function B_m^{-1} that breaks the super cells back to their components.

If $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ is a CA function over state set S then

$$B_m \circ G \circ B_m^{-1} : (S^m)^{\mathbb{Z}} \longrightarrow (S^m)^{\mathbb{Z}}$$

is also a CA function, the ***m*-block** presentation of G. Its state set is S^m.

Notation 2:

For any function $\alpha : A \longrightarrow B$ between two finite sets A and B, let

$$\tilde{\alpha} : A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$$

be the function between configurations that applies α at all cells independently: For all $c \in A^{\mathbb{Z}}$ and all $i \in \mathbb{Z}$,

$$\tilde{\alpha}(c)_i = \alpha(c_i).$$

Proposition. For every one-dimensional RCA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ there is a positive integer n such that the $2n$-block presentation of $G \circ \sigma^n$ is a radius-$\frac{1}{2}$ PCA (up to renaming the states).

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More precisely: There exist

- finite sets L and R (the sets of left and right stairs for G),
- a PCA
$$F : (R \times L)^{\mathbb{Z}} \longrightarrow (R \times L)^{\mathbb{Z}}$$
with the neighborhood $(0, 1)$ and the state set $R \times L$ (the bijection
$$\pi : R \times L \longrightarrow R \times L$$
applied at all cells being $\pi = \varphi \circ \psi$)
- number $n = 3r$,
- a state renaming bijection $\varphi : S^{2n} \longrightarrow R \times L$

such that

$$B_{2n} \circ \mathbf{G} \circ \sigma^{\mathbf{n}} \circ B_{2n}^{-1} = \tilde{\varphi}^{-1} \circ \mathbf{F} \circ \tilde{\varphi}.$$

Denoting by $H : (R \times L)^{\mathbb{Z}} \longrightarrow (R \times L)^{\mathbb{Z}}$ the function that shifts the L-components to the left:

$$\ldots, (r_{-1}, \ell_{-1}), (r_0, \ell_0), (r_1, \ell_1), \cdots \mapsto \ldots, (r_{-1}, \ell_0), (r_0, \ell_1), (r_1, \ell_2), \ldots$$

we have that

$$F = \tilde{\pi} \circ H = \tilde{\varphi} \circ \tilde{\psi} \circ H.$$

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The claim

$$B_{2n} \circ \mathbf{G} \circ \sigma^{\mathbf{n}} \circ B_{2n}^{-1} = \tilde{\varphi}^{-1} \circ \mathbf{F} \circ \tilde{\varphi}$$

can thus be written as

$$\mathbf{G} \circ \sigma^{\mathbf{3r}} = B_{6r}^{-1} \circ \tilde{\psi} \circ H \circ \tilde{\varphi} \circ B_{6r}.$$

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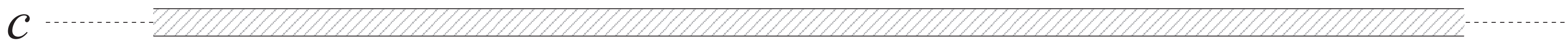

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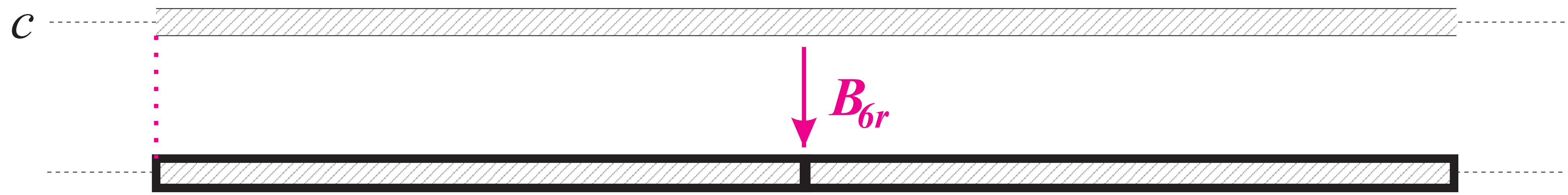

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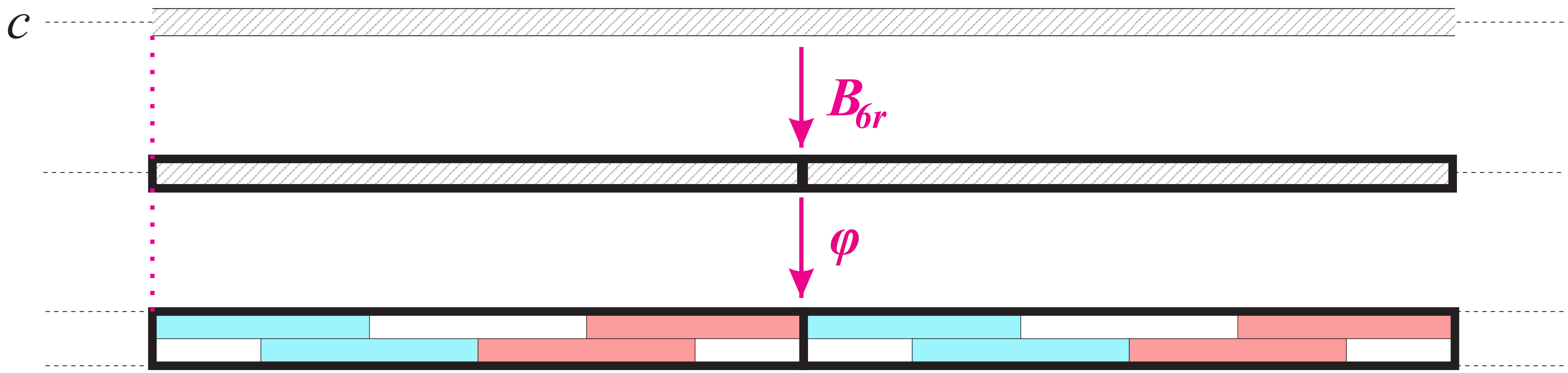

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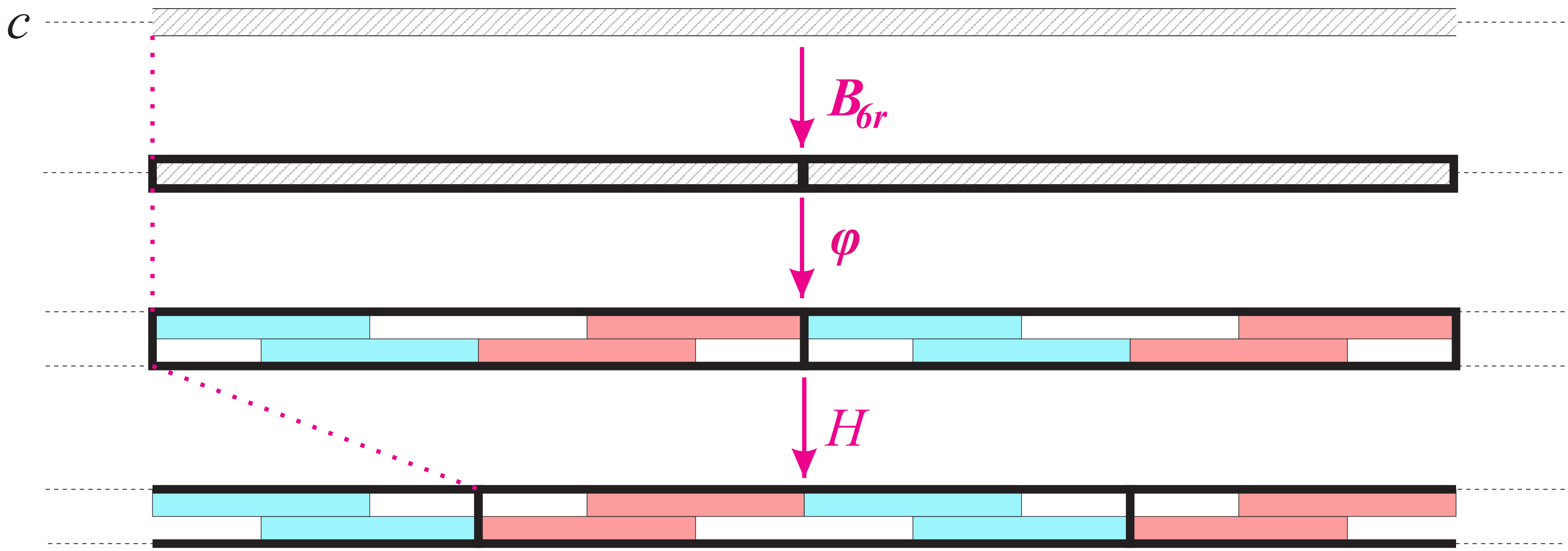

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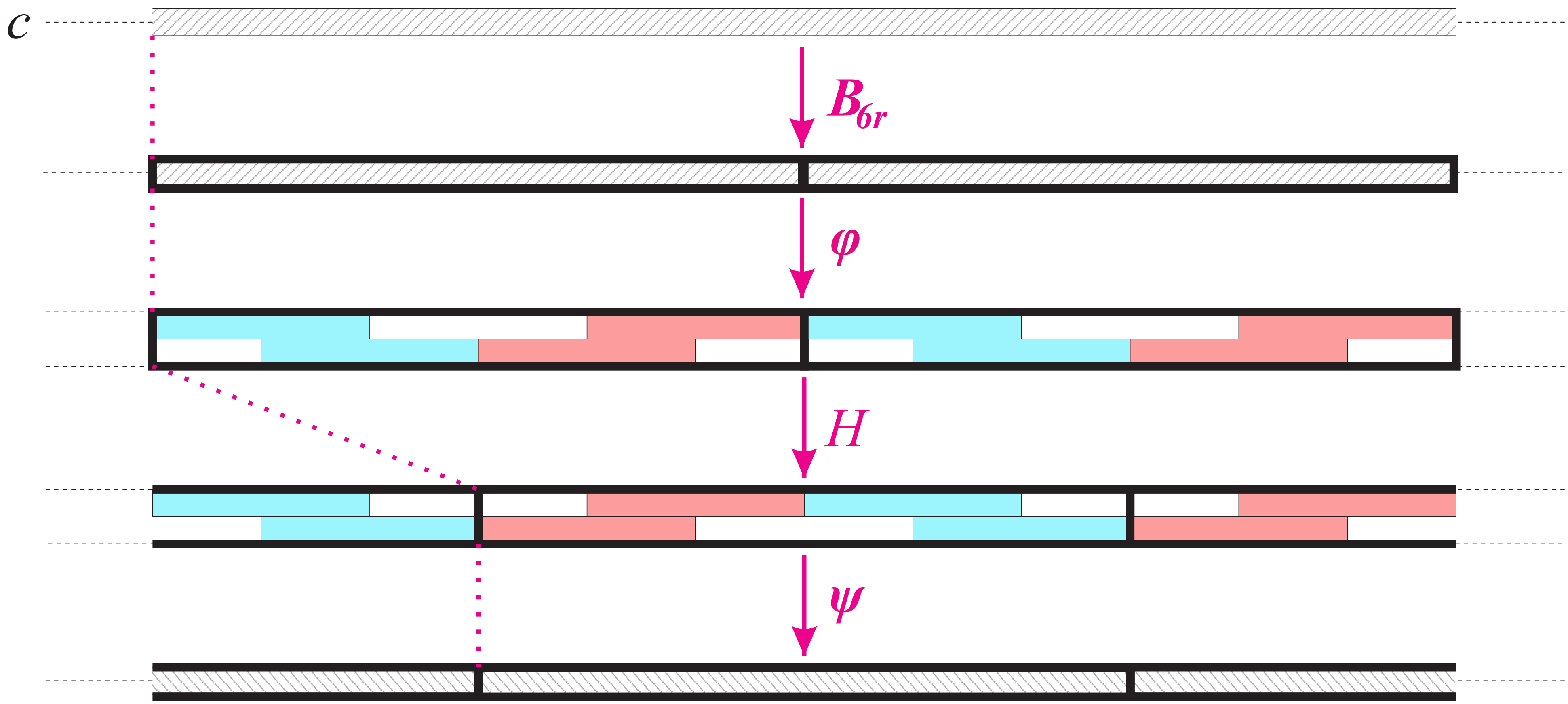

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