Conserved quantities

Let G be a d-dimensional CA function over state set S. An **additive quantity** is any function

$$\mu : S \longrightarrow \mathbb{R}$$

that assigns a real value to each state.

We'd like to extend μ to configurations by summing up the values assigned to states of cells, over all cells, and then say that μ is conserved by G if the sum remains unchanged when G is applied.

But the sum may be infinite!

Two solutions:

(1) Consider only q-finite configurations where q is a quiescent state satisfying $\mu(q) = 0$.

(2) Consider only strongly periodic configurations and make the sum over one period only.

First approach: Let q be a quiescent state of G and assume μ satisfies $\mu(q) = 0$.

For any q-finite configuration c define

$$\hat{\mu}_F(c) = \sum_{\vec{n} \in \mathbb{Z}^d} \mu(c(\vec{n})).$$

This sum has only a finite number of non-zero values.

We say that μ is **conserved on finite configurations** by G if and only if for all finite configurations c

$$\hat{\mu}_F(G(c)) = \hat{\mu}_F(c).$$

Second approach: For any strongly periodic configuration c define

$$\hat{\mu}_P(c) = \frac{1}{k^d} \sum_{\vec{n} \in C} \mu(c(\vec{n}))$$

where k is such that c is σ_i^k invariant for all $i = 1, 2, \ldots d$, and C is a hypercube of size k^d.

For any such k and C we obtain the same value of $\hat{\mu}_P(c)$. It is the **average** value in c over all cells.

We say that μ is **conserved on periodic configurations** by G if and only if for all strongly periodic configurations c

$$\hat{\mu}_P(G(c)) = \hat{\mu}_P(c).$$

The two approaches are in fact equivalent:

Proposition. Let G be a CA with quiescent state q and let μ be an additive quantity satisfying $\mu(q) = 0$. Then μ is conserved on periodic configurations if and only if μ is conserved on finite configurations.

Proof.

Remark: The restriction that $\mu(q) = 0$ is not important.

If μ is conserved on periodic configurations so is μ' where $\mu'(s) = \mu(s) + \alpha$ for any constant $\alpha \in \mathbb{R}$.

So if $\mu(q) \neq 0$ we can consider $\mu'(s) = \mu(s) - \mu(q)$ instead. It satisfies the constraint $\mu'(q) = 0$ and it is conserved on periodic configurations if and only if μ is.

Since the two approaches are equivalent we concentrate on finite configurations in the following. From now on we denote briefly $\hat{\mu}$ for $\hat{\mu}_F$, and say that the quantity is **conserved** if it is conserved on finite configurations.

If μ is conserved by G then swapping the state of a single cell should cause the same amount of change in $\hat{\mu}$ before and after the application of the CA:

Proposition. Quantity μ is conserved if and only if for all pairs of finite configurations c_1 and c_2 that differ in a single cell it holds that

$$\hat{\mu}(c_1) - \hat{\mu}(c_2) = \hat{\mu}(G(c_1)) - \hat{\mu}(G(c_2)).$$

Proof.