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- If $N = (\vec{n}_1, \ldots, \vec{n}_m)$ is the neighborhood of G, the right-hand-side becomes

$$\hat{\mu}(G(c_1)) - \hat{\mu}(G(c_2)) = \sum_{\vec{n} \in A} \mu(G(c_1)(\vec{n})) - \sum_{\vec{n} \in A} \mu(G(c_2)(\vec{n}))$$

  where

$$A = \{-\vec{n}_i \mid i = 1, 2, \ldots, m\}$$

  is the set of all cells that have $\vec{0}$ as a neighbor.

$$\hat{\mu}(c_1) - \hat{\mu}(c_2) \; = \; \mu(G(c_1)) - \mu(G(c_2)) \tag{$*$}$$

$$\mu(c_1(\vec{0})) - \mu(c_2(\vec{0})) \; = \; \sum_{\vec{n}\in A} \mu(G(c_1)(\vec{n})) - \sum_{\vec{n}\in A} \mu(G(c_2)(\vec{n})) \tag{$\#$}$$

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- Let
$$B = N(A) = \{\vec{n}_j - \vec{n}_i \mid i, j = 1, 2, \ldots, m\}$$
be the set of cells whose state influences a cell in A. If $c'|_B = c|_B$ then
$$\sum_{\vec{n}\in A} \mu(G(c')(\vec{n})) = \sum_{\vec{n}\in A} \mu(G(c)(\vec{n})).$$
So the right-hand-side of (#) only depends on the B-patterns of c_1 and c_2.
- Because $\vec{0} \in B$ also the left-hand-side of (#) only depends on the B-patterns of c_1 and c_2.

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- Because $\vec{0} \in B$ also the left-hand-side of (#) only depends on the B-patterns of c_1 and c_2.
- Thus it is enough to check (#) for all B-patterns that only differ at cell $\vec{0}$: for all $p_1, p_2 \in S^B$ such that $p_1(\vec{n}) \neq p_2(\vec{n}) \iff \vec{n} = \vec{0}$
  $$\mu(p_1(\vec{0})) - \mu(p_2(\vec{0})) = \sum_{\vec{n}\in A} \mu(G^{(B\to A)}(p_1)(\vec{n})) - \sum_{\vec{n}\in A} \mu(G^{(B\to A)}(p_2)(\vec{n})).$$

Conclusion: Quantity μ is conserved by G if and only if

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holds for all patterns $p_1, p_2 \in S^B$ that differ only in cell $\vec{0}$.

There are only **finitely many equations** to check! These can be checked effectively, implying an **algorithm** to check if a given quantity μ is conserved by a given CA G.

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Another equation $\mu(q) = 0$ may be included for quiescent q, but this is not necessary since $\mu(q) = 0$ only scales the quantity by an additive constant.

We assumed that G has a quiescent state q. If the CA has no quiescent state q we may pick an arbitrary ground state q and consider q-finite configurations. The considerations work unaltered, but we have to add the equation

$$\mu(q) = \mu(f(q, q, \ldots, q))$$

stating that the quantity is conserved on the q-uniform ground configuration.

A more careful analysis to reduce the number of equations.

Consider the 1D case. The quiescent configuration can be transformed into any finite configuration by swapping states of cells one-by-one in the left-to-right order. So it is sufficient that

$$\hat{\mu}(c_1) - \hat{\mu}(c_2) = \hat{\mu}(G(c_1)) - \hat{\mu}(G(c_2)) \qquad (*)$$

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- $c_1(n) = c_2(n)$ for all $n < 0$.
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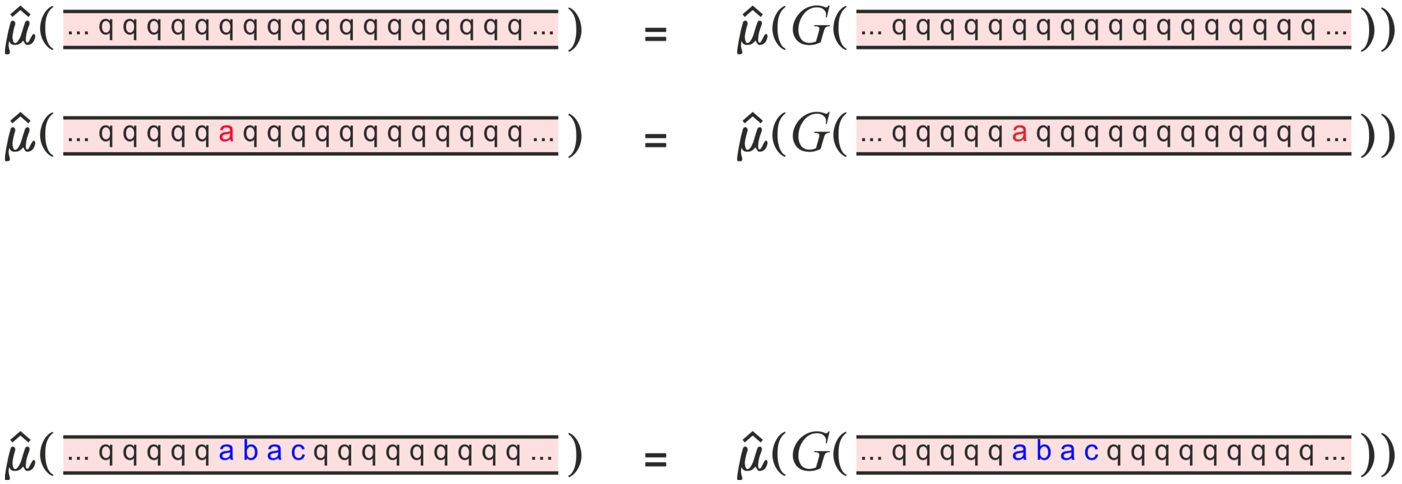

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Example. Let us find all conserved μ of the traffic CA (=elementary CA 226). The local rule replaces pattern 01 by pattern 10:

$$\begin{array}{cccc} 000 \mapsto 0 & 001 \mapsto 1 & 010 \mapsto 0 & 011 \mapsto 0 \\ 100 \mapsto 0 & 101 \mapsto 1 & 110 \mapsto 1 & 111 \mapsto 1 \end{array}$$

Example. Let us find all conserved μ of the 1D CA with the radius-$\frac{1}{2}$ neighborhood $N = (0, 1)$, the state set is $S = \{0, 1, 2\}$ and the local rule

$$f(a, b) = \begin{cases} 2, & \text{if } a = 2, \\ 0, & \text{if } a \neq 2 \text{ and } a + b \text{ is even, and} \\ 1, & \text{if } a \neq 2 \text{ and } a + b \text{ is odd.} \end{cases}$$

Remark: For any fixed CA, the additive quantities that it conserves form a **linear space**, when sum and scalar product are defined in the natural way (=point wise):

$$\begin{array}{rcll} (\mu+\mu')(a) & = & \mu(a)+\mu'(a) & \text{for all } \mu,\mu' \in \mathbb{R}^S \text{ and } a \in S, \\ (r\mu)(a) & = & r\mu(a) & \text{for all } \mu \in \mathbb{R}^S,\ r \in \mathbb{R} \text{ and } a \in S. \end{array}$$

Namely, if G conserves μ and μ' then for any q-finite configuration c holds

$$(\mu+\mu')(c) = \mu(c)+\mu'(c) = \mu(G(c))+\mu'(G(c)) = (\mu+\mu')(G(c))$$

so also $\mu+\mu'$ s conserved. And for every $r \in \mathbb{R}$ and c

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In fact our equations to find conserved μ are linear, so we effectively made a **system of linear equations** whose solutions are exactly the conserved quantities of G.

Cellular automata dynamical systems

We endow the configuration space $S^{\mathbb{Z}^d}$ with a **metric** under which CA functions G are continuous.

Define the distance of configurations $c \neq e$ as

$$d(c, e) = 2^{-\min\{\|\vec{x}\| \mid c(\vec{x}) \neq e(\vec{x})\}}$$

where we use the notation

$$\|(x_1, \ldots, x_d)\| = \max\{|x_1|, \ldots, |x_d|\}.$$

(And for $c = e$ the distance $d(c, e) = 0$.)

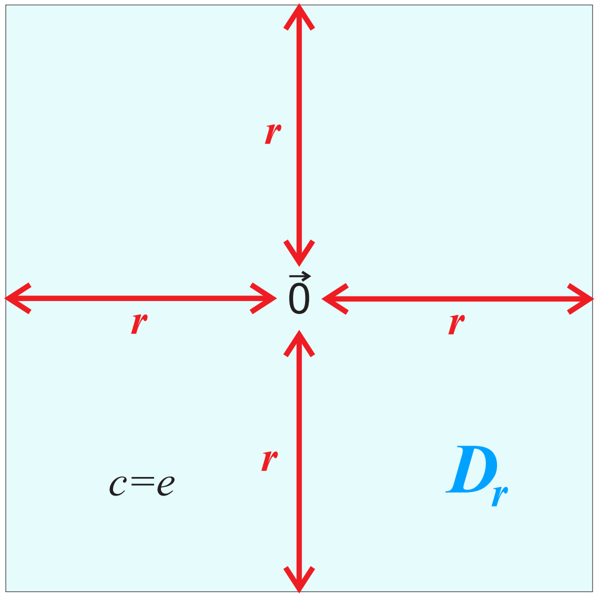
$$d(c,e) = 2^{-\min\{\|\vec{x}\| \ \mid \ c(\vec{x}) \neq e(\vec{x})\}}$$

So

$$d(c,e) < 2^{-r} \iff c_{|D_r} = e_{|D_r}$$

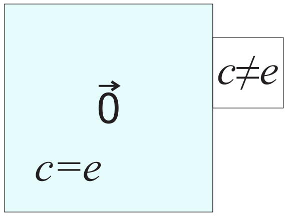
where

$$D_r = \{\vec{x} \in \mathbb{Z}^d \mid \|\vec{x}\| \leq r\}.$$



$$d(c,e) = 2^{-\min\{\|\vec{x}\| \mid c(\vec{x}) \neq e(\vec{x})\}}$$

Two configurations c and e are close (i.e., $d(c,e)$ is small) if c and e agree on a large region around the origin.



$c{\neq}e$

$\vec{0}$

$c{=}e$

$d(c,e)$ large if c and e differ close to $\vec{0}$.

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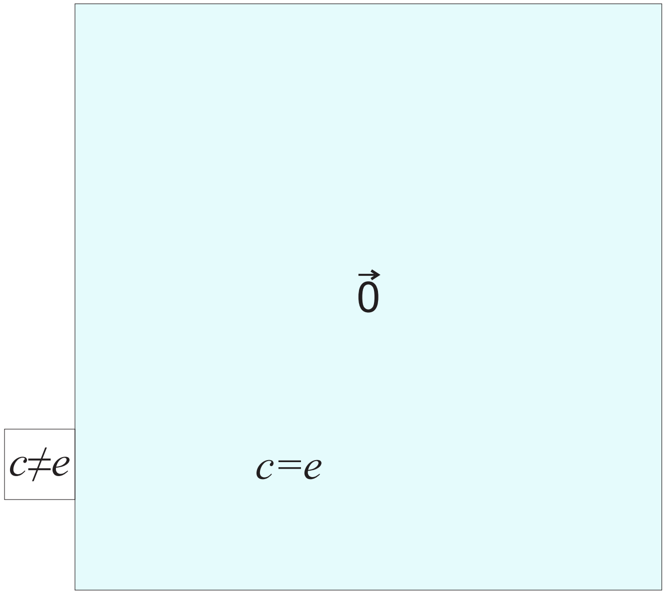
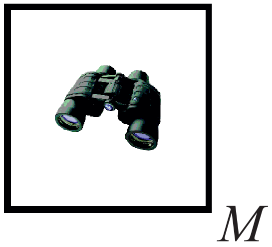


$d(c,e)$ small if c and e agree in a large region around $\vec{0}$.

Finite set $M \subseteq \mathbb{Z}^d$ is an observation window that corresponds to a ”measuring device”. Two configurations c and e seem identical through the measuring device if $e_{|M} = c_{|M}$.



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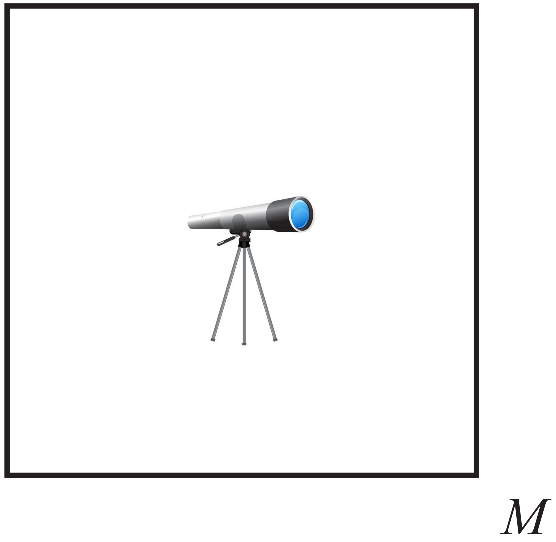

Larger window M means better accuracy of observation.

Recall the definition of a metric space: (X, d) is a metric space if $X \neq \emptyset$ is a set and

$$d : X \times X \longrightarrow \mathbb{R}$$

is a distance function that satisfies the following three conditions:

(i) $d(x, y) > 0$ for $x \neq y$, and $d(x, y) = 0$ for $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Many essential properties of the space can be proved using the axioms (i)–(iii) only.

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Lemma. The configuration space $X = S^{\mathbb{Z}^d}$ with the distance function

$$d(c,e) = 2^{-\min\{\|\vec{x}\| \ | \ c(\vec{x}) \neq e(\vec{x})\}}$$

is a metric space.

In fact: The space is an **ultrametric** as it satisfies the strong triangle inequality

(iii') $d(x,y) \leq \max\{d(x,z), d(z,y)\}$.

Proof.

Let (X, d) be a metric space.

For every $\varepsilon > 0$ and $x \in X$ we denote

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

and call $B_\varepsilon(x)$ the (open) **ε-ball** with center x.

A set $U \subseteq X$ is **open** if

$$\forall x \in U, \ \exists \varepsilon > 0 \ : \ B_\varepsilon(x) \subseteq U.$$

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

$$U \text{ is open} \iff \forall x \in U,\ \exists \varepsilon > 0\ :\ B_\varepsilon(x) \subseteq U.$$

Proposition. Let (X, d) be a metric space. Then

(i) $\emptyset$ and X are open,

(ii) arbitrary unions of open sets are open, and

(iii) intersections of $\underline{\text{finitely}}$ many open sets are open.

Proof.

Corollary. A set is open if and only if it is a union of open balls.

Proof.

(i) $\emptyset$ and X are open,

(ii) arbitrary unions of open sets are open, and

(iii) intersections of finitely many open sets are open.

Many properties of metric spaces can be proved using properties (i), (ii) and (iii) only.

Further abstraction: A pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a family of subsets of X is a **topological space**, family $\mathcal{T}$ is called a **topology** on X, and sets in $\mathcal{T}$ are called **open** if axioms (i), (ii) and (iii) are satisfied.

Thus the family of open sets of a metric space (X, d) forms a topology on X. It is called a **metric topology**. There are also topologies that are not metrizable, i.e., not defined by any metric.

Consistently with metric spaces we define:

A set is **closed** if its complement is open

A set is **clopen** if it is both open and closed.

By de Morgan's laws closed sets behave dually to open sets:

Proposition. Let $(X, \mathcal{T})$ be a topological space.

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A quick note on balls in metric spaces

- **An open ball**

  $$B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}$$

  is open in the topology.

- **A closed ball**

  $$\overline{B}_\varepsilon(x) = \{y \mid d(x,y) \leq \varepsilon\}$$

  is closed in the topology.

Further terminology: Let $(X, \mathcal{T})$ be a top. space.

- $x \in X$ is **isolated** if $\{x\}$ is open. A space is **perfect** if it has no isolated points.

- Let $A \subseteq X$. The **closure** of A is

$$\overline{A} = \bigcap_{\substack{F closed \\ A \subseteq F}} F.$$

It is the smallest closed set that contains A:

$$F \text{ closed}, A \subseteq F \implies \overline{A} \subseteq F.$$

- Set $A \subseteq X$ is **dense** if $\overline{A} = X$.

- Dual to closure: The **interior** of A is

$$A^\circ = \bigcup_{\substack{V open \\ V \subseteq A}} V.$$

It is the largest open subset of A:

$$V \text{ open}, V \subseteq A \implies V \subseteq A^\circ.$$

Convergence of sequences

A sequence $x_1, x_2, \ldots$ **converges** to x if for every open neighborhood U of x there is $n \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq n$.

In the metric setting: For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_i, x) < \varepsilon$ for all $i \geq n$.

Proposition. In a metric space every converging sequence converges to a unique point.

Proof.

We denote the unique limit by $\lim_{i\to\infty} x_i$.

Base of a topology

A family $\mathcal{B} \subseteq \mathcal{T}$ is a **base** of topology $\mathcal{T}$ iff every open set is a union of some members of $\mathcal{B}$.

Example. In a metric space (X, d) open sets are precisely unions of open balls. Thus the family

$$\{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$$

of all open balls is a base.

Proposition. A family $\mathcal{B} \subseteq \mathcal{T}$ is a base of topology $\mathcal{T}$ if and only if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \;:\; x \in B \subseteq U.$$

Proof.

Compactness

Let $\mathcal{T}$ be a topology on X, and let $A \subseteq X$.

A family $\mathcal{U} \subseteq \mathcal{T}$ is called an **open cover** of A if

$$A \subseteq \bigcup_{V \in \mathcal{U}} V.$$

A subfamily $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathcal{U}$ is called a **subcover** if it is also a cover of A.

Set $A \subseteq X$ is called **compact** if every open cover of A has a finite subcover of A. The topology is called compact if the whole space X is compact.

In other words: a topology is compact iff every family of open sets whose union is X has a finite subfamily whose union is X.

Compactness of X could as well be defined using a dual concept:

Proposition. Topology of X is compact if and only if every family of closed sets whose intersection is empty has a finite subfamily whose intersection is empty.

Corollary. Let

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

be an infinite chain of closed sets in a compact space X. If $F_i \neq \emptyset$ for all i then

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

Compactness in metric spaces is equivalent to **sequential compactness**:

Proposition. Let (X, d) be a metric space. Set $A \subseteq X$ is compact if and only if every sequence of elements of A has a subsequence that converges to an element of A

(A **subsequence** of a sequence $x_1, x_2, \ldots$ is a sequence $x_{i_1}, x_{i_2}, \ldots$ for some $i_1 < i_2 < \ldots$.)

Proof

In compact metric spaces compact sets are exactly the closed sets:

Proposition. Let X be a **compact** metric space. For $A \subseteq X$

$$A \text{ closed } \iff A \text{ compact.}$$

Proof.

Baire property

Set $A \subseteq X$ is **residual** if it is the intersection of countably many dense open sets. A topological space X is a **Baire space** if every residual set is dense.

That is: in a Baire space, if $U_1, U_2, \ldots$ are open sets such that $\overline{U_i} = X$ for all i then also $\overline{A} = X$ where

$$A = \bigcap_{i=1}^{\infty} U_i.$$

Example. Set $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is **not** a Baire space:

For every $q \in \mathbb{Q}$ the set $\mathbb{Q} \setminus \{q\}$ is open and dense, but the (countable) intersection

$$\bigcap_{q \in \mathbb{Q}} \mathbb{Q} \setminus \{q\}$$

is empty.

Baire property

Set $A \subseteq X$ is **residual** if it is the intersection of countably many dense open sets. A topological space X is a **Baire space** if every residual set is dense.

That is: in a Baire space, if $U_1, U_2, \ldots$ are open sets such that $\overline{U}_i = X$ for all i then also $\overline{A} = X$ where

$$A = \bigcap_{i=1}^{\infty} U_i.$$

Proposition. Every compact metric space is a Baire space.

Proof.