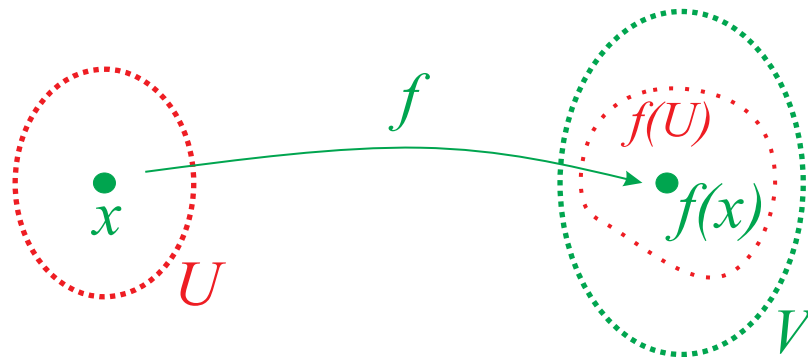


# Continuity

Let  $X, Y$  be topological spaces. Function  $f : X \longrightarrow Y$  is **continuous at point**  $x \in X$  if

$$\begin{aligned} V \subseteq Y \text{ open, } f(x) \in V \\ \implies \exists \text{ open } U \subseteq X : x \in U \text{ and } f(U) \subseteq V. \end{aligned}$$

(For every open neighborhood  $V$  of  $f(x)$  there exists an open neighborhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .)



Function  $f : X \longrightarrow Y$  is **continuous** if it is continuous at every  $x \in X$ .

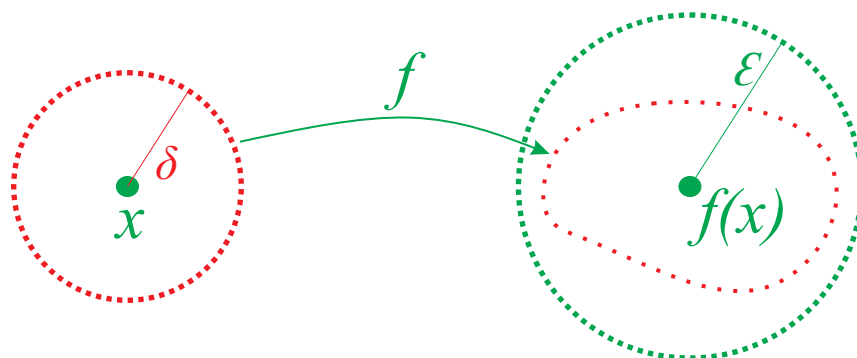
**Proposition.** The following conditions are equivalent:

- (i) Function  $f : X \longrightarrow Y$  is continuous,
- (ii) pre-image  $f^{-1}(V)$  is open for each open  $V \subseteq Y$ ,
- (iii) pre-image  $f^{-1}(F)$  is closed for each closed  $F \subseteq Y$ .

**Proof.**

In the metric case:  $f$  is continuous if

$$\forall \varepsilon > 0, \forall x \in X, \exists \delta > 0 : f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$



Number  $\delta$  may depend on point  $x$ . If  $\delta$  can be chosen independently of  $x$  then function  $f$  is **uniformly continuous**:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X : f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$

Every uniformly continuous function is continuous. In the compact cases also the converse holds:

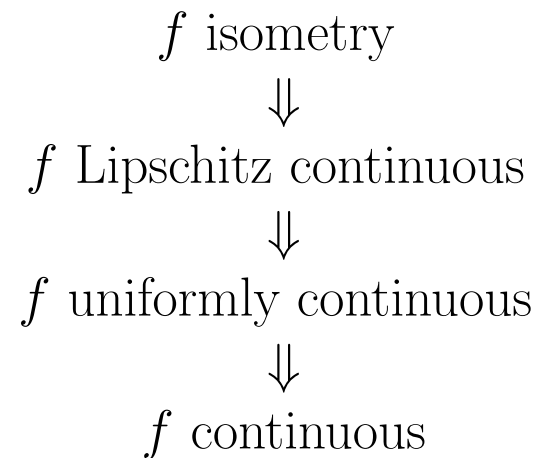
**Proposition.** Let  $X$  and  $Y$  be compact metric spaces. Function  $f : X \longrightarrow Y$  is continuous if and only if it is uniformly continuous.

If there exists a positive constant  $r$  such that  $\delta = r \cdot \varepsilon$  works for all  $x$  and  $\varepsilon$ , then  $f$  is **Lipschitz continuous**.

It is an **isometry** if distances remain unchanged:

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Clearly:



**Note:** Continuity is defined for all topological spaces; the other three concepts only for metric spaces.

**Proposition.** Let  $X$  be a metric space and  $Y$  a topological space. A function

$$f : X \longrightarrow Y$$

is continuous if and only if for every converging sequence  $x_1, x_2, \dots$  the sequence  $f(x_1), f(x_2), \dots$  converges and

$$\lim_{i \rightarrow \infty} f(x_i) = f\left(\lim_{i \rightarrow \infty} x_i\right).$$

**Proof.**

**Proposition.** Let  $f : X \longrightarrow Y$  be continuous. For every compact  $A$  the set  $f(A)$  is compact.

**Proof.**

**Proposition.** Let  $f : X \longrightarrow Y$  be a continuous bijection between compact metric spaces  $X$  and  $Y$ . Then the inverse function  $f^{-1} : Y \longrightarrow X$  is also continuous.

**Proof.**

If  $f : X \longrightarrow Y$  is a bijection and both  $f$  and  $f^{-1}$  are continuous then  $f$  is a **homeomorphism** and spaces  $X$  and  $Y$  are **homeomorphic**. This is the “isomorphism” of topological structures.

**Corollary.** Continuous bijection between compact metric spaces is a homeomorphism.

## Metric on $A^{\mathbb{Z}^d}$

The distance of configurations  $c \neq e$  is

$$d(c, e) = 2^{-\min\{\|\vec{x}\| \mid c(\vec{x}) \neq e(\vec{x})\}}$$

where we use the notation

$$\|(x_1, \dots, x_d)\| = \max\{|x_1|, \dots, |x_d|\}.$$

For any  $r \in \mathbb{R}$  and  $c, e \in A^{\mathbb{Z}^d}$  we have

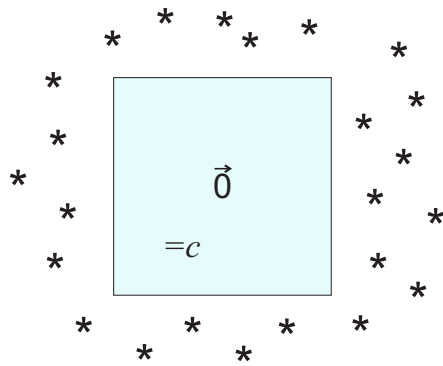
$$d(c, e) < 2^{-r} \iff e(x_1, \dots, x_d) = c(x_1, \dots, x_d) \text{ when } |x_i| \leq r \text{ for all } i.$$

The **open ball** of radius  $\varepsilon = 2^{-r}$  centered at  $c \in A^{\mathbb{Z}^d}$  is then

$$B_\varepsilon(c) = \{e \in A^{\mathbb{Z}^d} \mid e(x_1, \dots, x_d) = c(x_1, \dots, x_d) \text{ when } |x_i| \leq r \text{ for all } i\}.$$

$$B_\varepsilon(c) = \{e \in A^{\mathbb{Z}^d} \mid e(x_1, \dots, x_d) = c(x_1, \dots, x_d) \text{ when } |x_i| \leq r \text{ for all } i\}.$$

The ball consists of all configurations  $e$  that agree with  $c$  inside the hypercube  $D = \{-r, \dots, r\} \times \dots \times \{-r, \dots, r\}$ :



Thus open balls are precisely sets defined by finite patterns  $p \in A^D$  for  $D = \{-r, \dots, r\} \times \dots \times \{-r, \dots, r\}$ :

$$\{c \in A^{\mathbb{Z}^d} \mid c|_D = p\}.$$

Open balls:

$$\{c \in A^{\mathbb{Z}^d} \mid c|_D = p\}$$

for  $p \in A^D$  with some  $D = \{-r, \dots, r\} \times \dots \times \{-r, \dots, r\}$ .

Recall that open balls are a base of the topology: open sets are precisely the unions of open balls.

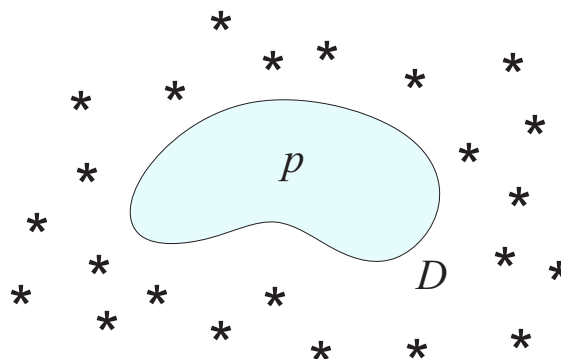
More generally we define a **cylinder** determined by a finite pattern  $p \in A^D$  with any finite domain  $D \subseteq \mathbb{Z}^d$  as the set of all configurations that have pattern  $p$  in domain  $D$ :

$$[p] = \{e \in A^{\mathbb{Z}^d} \mid e|_D = p\}.$$

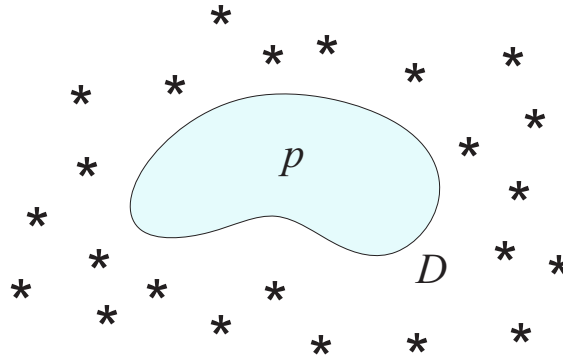
We also denote this by

$$\text{Cyl}(c, D)$$

for any  $c \in [p]$ .



$$[p] = \{e \in A^{\mathbb{Z}^d} \mid e|_D = p\}$$



- Open balls are cylinders (where  $D$  is a square centered at  $\vec{0}$ ),
- Every cylinder is a union of open balls: Indeed, if  $D \subseteq E$  and  $p \in A^D$  then

$$[p] = \bigcup_{\substack{q \in A^E \\ q|_D = p}} [q].$$

Thus cylinders form a **basis** of the topology.

**Equivalently:** A set  $U \subseteq A^{\mathbb{Z}^d}$  is open iff for every  $c \in U$  there exists a finite  $D \subseteq \mathbb{Z}^d$  such that  $[c|_D] \subseteq U$ .

Cylinders are also closed because their complements are open:

$$A^{\mathbb{Z}^d} \setminus [p] = \bigcup_{\substack{q \in A^D \\ q \neq p}} [q].$$

Thus cylinders are a clopen basis of the topology.

**Remark.** Our **metric** is **not translation invariant**: difference at the center cell makes two configurations more distant from each other than a difference at a cell far from the center.

**However,** the **topology** that the metric induces is **translation invariant**: Translations of cylinders are cylinders so the base (=the set of cylinders) is invariant under translations. In the topology the center cell is not more important than any other cell.

**Another remark.** The same topology (homeomorphic to the **Cantor space**) is induced by many other metrics. If the open balls are cylinders, and if all cylinders are subsets of some open balls then the metric defines the same topology.

**For example,** we can define a metric by

$$d(c, e) = f(\min\{g(\vec{x}) \mid c(\vec{x}) \neq e(\vec{x})\})$$

where  $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$  is any decreasing function such that

$$\lim_{n \rightarrow \infty} f(n) = 0,$$

and the function  $g : \mathbb{Z}^d \longrightarrow \mathbb{R}_+$  can be any function such that for every  $a$  there are only finitely many  $\vec{x} \in \mathbb{Z}^d$  with the property  $g(\vec{x}) < a$ .

This always defines the same topology as our choice

$$f(x) = 2^{-x}, \quad g(x_1, \dots, x_d) = \max\{|x_1|, \dots, |x_d|\}.$$

## Yet another remark.

- The set of strongly periodic configurations is a countable dense subset of  $A^{\mathbb{Z}^d}$ .
- The set of cylinders is a countable base of the topology.

## Proof.

**Earlier:** we said that a sequence  $c_1, c_2, \dots$  of configurations **converges** to  $c \in A^{\mathbb{Z}^d}$  if

$$\forall \vec{x} \in \mathbb{Z}^d, \exists N \in \mathbb{N}, \forall k > N : c_k(\vec{x}) = c(\vec{x}).$$

This concept of convergence is identical to convergence in our metric:

$\Leftarrow$  Suppose  $c_1, c_2, \dots$  converges to  $c$  under the metric.

$\Rightarrow$  Suppose  $c_1, c_2, \dots$  converges to  $c$  in the earlier sense.

We have proved that every sequence has a converging subsequence.

We have proved that sequential compactness implies compactness (when there is a countable base).

**Corollary.** The metric space  $A^{\mathbb{Z}^d}$  is compact.

We have also proved the following property for any CA  $G : A^{\mathbb{Z}^d} \longrightarrow A^{\mathbb{Z}^d}$  and for any converging sequence  $c_1, c_2, \dots$  of configurations:

**The sequence  $G(c_1), G(c_2), \dots$  converges and**

$$\lim_{i \rightarrow \infty} G(c_i) = G(\lim_{i \rightarrow \infty} c_i).$$

But this means continuity:

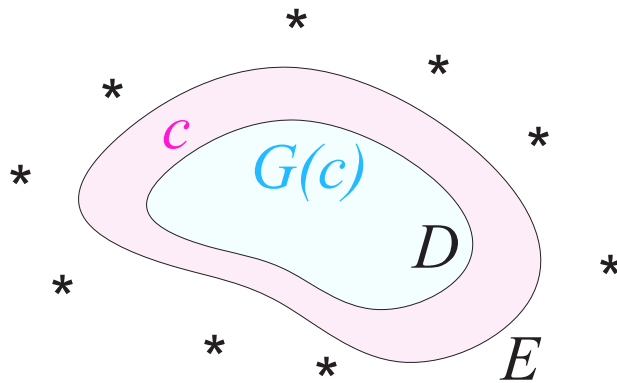
**Corollary.** Every CA function  $G : A^{\mathbb{Z}^d} \longrightarrow A^{\mathbb{Z}^d}$  is continuous.

**Alternatively,** continuity of a cellular automaton  $G : A^{\mathbb{Z}^d} \rightarrow A^{\mathbb{Z}^d}$  can be seen as follows:

For every configuration  $c$  and every finite  $D \subseteq \mathbb{Z}^d$  there exists a finite  $E \subseteq \mathbb{Z}^d$  such that

$$G([c|_E]) \subseteq [G(c)|_D].$$

(As  $E$  one can choose any finite set that contains all neighbors of all cells in  $D$ .)



For every open set  $U$  containing  $G(c)$  there is a finite  $D$  such that  $[G(c)|_D] \subseteq U$ , so there is an open set  $[c|_E]$  containing  $c$  that is mapped into  $U$ .

Pairs  $(X, F)$  where  $X$  is a compact metric space and  $F : X \longrightarrow X$  is continuous are commonly called (topological) **dynamical systems**.

So  $(S^{\mathbb{Z}^d}, G)$  is a (topological) dynamical system for each CA function  $G$ .

# Hedlund's theorem

Every CA function

- is **continuous**, and
- **commutes with translations**.

Hedlund's theorem states that also the converse is true: Any function  $S^{\mathbb{Z}^d} \rightarrow S^{\mathbb{Z}^d}$  that is continuous and commutes with translations is a CA function.

Basically:

- **continuity**  $\implies$  each cell has a local rule,
- **translation invariance**  $\implies$  the local rules are the same.

**Proposition.** Function  $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$  is a CA function if and only if it is continuous and it commutes with all translations of  $\mathbb{Z}^d$ .

**Proof.**

We have the following corollary that was previously proved as a proposition:

**Corollary.** If cellular automaton  $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$  is bijective then its inverse function  $G^{-1}$  is also a cellular automaton.

**Proof.** Enough to note that  $G^{-1}$  is continuous and commutes with translations.

We can also easily re-prove some of our earlier propositions:

**Proposition.** If cellular automaton  $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$  is surjective among strongly periodic configurations then  $G$  is surjective.

**Proof.** Follows from the facts that  $G(S^{\mathbb{Z}^d})$  is compact (as the image of a compact set under a continuous function) and therefore closed, and that the set of strongly periodic configurations is dense.

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The same proof works also with “**finite**” in place of “**strongly periodic**”.