Limit sets

The **limit set** of a CA G is

$$\Omega_G = \bigcap_{n=0}^{\infty} G^n(S^{\mathbb{Z}^d}).$$

This means that $c \in \Omega_G$ if and only if c is not a Garden-of-Eden of G^n for any n.

Examples.

- The limit set of a surjective CA is the full configuration space $S^{\mathbb{Z}^d}$.
- The limit set of a nilpotent CA is the singleton set containing only the quiescent configuration.

Example. Let G be the elementary CA number 128. (The quiescent state 0 is spreading: $111 \mapsto 1$ and $abc \mapsto 0$ in all other cases.)

Patterns $100\ldots01$ are orphans of G^n for sufficiently large n
$\Longrightarrow$ no such pattern in any configuration of Ω_G.

In the remaining possibilities 1's form a contiguous segment:

$$\begin{gathered}\ldots 111111 \ldots\\ \ldots 000000 \ldots\\ \ldots 000111 \ldots\\ \ldots 111000 \ldots\\ \ldots 000111 \ldots 111000 \ldots\end{gathered}$$

Each such configuration has a pre-image of the same form
$\Longrightarrow$ they all belong to Ω_G.

A set $X \subseteq S^{\mathbb{Z}^d}$ of configurations is called a **subshift** if

- X is topologically closed, and
- X is translation invariant: $\tau(X) = X$ for all translations τ.

(Sometimes subshifts are also required to be non-empty.)

Example. For every CA G the set $G(S^{\mathbb{Z}^d})$ of non-GOE configurations is a subshift.

Proposition. Let G be a CA function and let Ω be its limit set. Then

(a) Ω is non-empty and a subshift (=closed and shift invariant).

(b) $G(\Omega) = \Omega$, and if $X \subseteq S^{\mathbb{Z}^d}$ satisfies $G(X) = X$ then $X \subseteq \Omega$.

(c) $c \in \Omega \iff$ there exist configurations $\ldots, c_{-2}, c_{-1}, c_0$ such that
$c_0 = c$ and $G(c_i) = c_{i+1}$ for all $i < 0$.

(d) Ω is finite $\iff$ G is nilpotent. In this case $|\Omega| = 1$.

Proof.

Attractor

A set $X \subseteq S^{\mathbb{Z}^d}$ is **inward** for CA G if

$$G(\overline{X}) \subseteq X^\circ.$$

A clopen set U is then inward iff $G(U) \subseteq U$.

Non-empty $A \subseteq S^{\mathbb{Z}^d}$ is an **attractor** of G if

$$A = \bigcap_{n=0}^{\infty} G^n(X)$$

for some inward set $X \subseteq S^{\mathbb{Z}^d}$.

It turns out (homework) that in the definition X can be chosen clopen: Non-empty $A \subseteq S^{\mathbb{Z}^d}$ is an attractor iff

$$A = \bigcap_{n=0}^{\infty} G^n(U)$$

for some **inward clopen** $U \subseteq S^{\mathbb{Z}^d}$. We say that A is the attractor determined by U.

$$A = \bigcap_{n=0}^{\infty} G^n(U)$$

Example. The limit set Ω is the attractor determined by $U = S^{\mathbb{Z}^d}$. Clearly Ω is the unique maximal attractor: $A \subseteq \Omega$ for all attractors A.

$$A = \bigcap_{n=0}^{\infty} G^n(U)$$

- Attractor is compact and contains a strongly periodic configuration.
- If A is an attractor then $G(A) = A$.
- The number of attractors is countable (since there are only countably many different clopen sets).

Proposition. Let A and B be two attractors of G.

(a) The union $A \cup B$ is an attractor.

(b) If $A \cap B \neq \emptyset$ then there exists an attractor $C \subseteq A \cap B$.

Proof.