Cellular Automata. Homework 11 (13.4.2026)

1. Prove that a configuration is in the limit set of the traffic CA if and only if it does not contain consecutive 1's to the right of consecutive 0's, that is, no pattern

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appears in c. (Traffic CA is the elementary CA number 226.)

2. Determine the limit set of the majority CA (elementary CA number 232).

3. CA G is called *stable* if there exists $n \geq 0$ such that

$$\Omega_G = G^n(S^{\mathbb{Z}^d}),$$

where Ω_G is the limit set. Determine which of the following cellular automata are stable.

   (a) Elementary CA 128 (see Example 38, page 107).

   (b) Xor CA (=Elementary CA 102).

   (c) Majority CA. (=Elementary CA 232).

4. See the previous problem for the definition of stable CA.

   (a) Show that all surjective CA are stable, and that all eventually periodic CA are stable.

   (b) Prove that it is undecidable if a given one-dimensional CA is stable.

5. Prove that if the limit set of a CA is a subshift of finite type (SFT) then the CA is stable. (Recall that a subshift of finite type is defined by forbidding a finite number of finite patterns. In other words, SFT is the complement of $\bigcup_\tau \tau(C)$ for some clopen set C, where the union is over all translations.)

6. Let G be the majority CA, and let

$$U = \{c \in \{0,1\}^{\mathbb{Z}} \mid c(0) = c(1) \text{ or } c(0) = c(-1)\}.$$

   (a) Verify that U is clopen, non-empty and satisfies $G(U) \subseteq U$.

   (b) Find the attractor A determined by U.

   (c) Determine the basin of attraction for the attractor you found in (b).

7. A subset $X \subseteq S^{\mathbb{Z}^d}$ is called *inward* for CA G if $G(\overline{X}) \subseteq X^\circ$, where $\overline{X}$ and X° are the closure and the interior of X, respectively, that is, $\overline{X}$ is the intersection of all closed supersets of X and X° is the union of all open subsets of X.

   (a) Prove that a clopen U is inward if and only if $G(U) \subseteq U$.

   (b) Prove that if X is inward then there exists an inward clopen U such that

$$G(X) \subseteq U \subseteq X$$

   (c) Prove that if

$$A = \bigcap_{n=0}^{\infty} G^n(X)$$

   for some inward set X then

$$A = \bigcap_{n=0}^{\infty} G^n(U)$$

   for some inward clopen set U.

(Note: Attractors are usually defined using arbitrary inward sets. This exercise shows that our definition based on clopen inward sets is equivalent.)