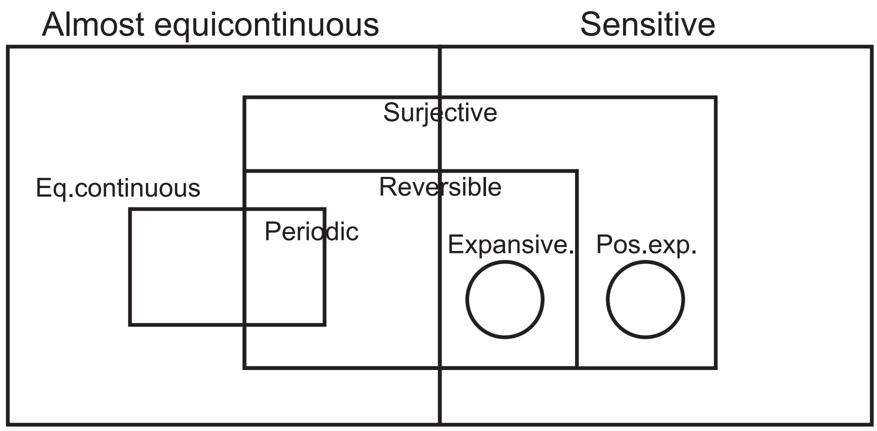
Cellular Automata. Homework 13 (27.4.2026)

1. Determine the exact position of the following one-dimensional CA in the sensitivity diagram



   (a) The majority CA (elementary CA number 232),
   (b) The shifted majority $G \circ \sigma$ where G is the majority CA and σ is the left shift,
   (c) Elementary CA number 106: $f(a, b, c) \equiv ab + c \pmod 2$.

2. Determine the exact position in the sensitivity diagram above of the six-state CA that multiplies by 3 in base 6, defined and analyzed in Problem 3, Homework set 8, and in Problem 1, homework set 9. Recall that this CA has state set $S = \{0, 1, 2, 3, 4, 5\}$, neighborhood vector $(0, 1)$, and the local rule $f : S^2 \longrightarrow S$ is given by
$$f(a, b) = 3 \cdot \text{parity}(a) + \left\lfloor \frac{b}{2} \right\rfloor,$$
where parity(a) is 0 or 1 if a is even or odd, respectively.

3. Let G be a one-dimensional CA with a strictly one-sided neighborhood vector $(1, 2, 3, \dots m)$.
   (a) Prove that G is either nilpotent or sensitive.
   (b) Prove that if G is reversible and the neighborhood of its inverse CA is $\{-n, -n+1, \dots, -1\}$ for some $n > 0$ then G is expansive.

4. Prove that a reversible CA G is sensitive if and only if its inverse G^{-1} is sensitive.

5. A one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ is *left-closing* if $G(c) \neq G(e)$ for any positively asymptotic configurations $c, e \in S^{\mathbb{Z}}$ such that $c \neq e$.
   (a) Prove that positively expansive CA are left-closing and that left-closing CA are surjective.
   (b) Prove that both implications in (a) are strict: show that there is a left-closing CA that is not positively expansive, and a surjective CA that is not left-closing. (Hint: Example 11 in the notes may be useful.)
   (c) Show that there is an algorithm to test if a given one-dimensional cellular automaton is left-closing. (Hint: what kind of path in the pair graph proves that the CA is not left-closing?)

6. For each of the following decision problems, prove that the problem or its complement is semi-decidable:
   (a) Given a one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$, is G equicontinuous ?
   (b) Given a one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$, a word $w \in S^*$ and a number m, is w an m-blocking word for G ?
   (c) Given a one-dimensional CA $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$, is G positively expansive ?

7. Consider a one-dimensional left-permutive CA with neighborhood $(n, n+1, \dots, n+m)$ where $n \neq 0$. (Recall: left-permutivity means that the local rule f satisfies
$$a \neq b \Longrightarrow f(a, a_1, \dots, a_m) \neq f(b, a_1, \dots, a_m),$$
for all $a_1, \dots, a_m$.) Prove that the CA is mixing.